ON THE CLASSICAL DIRICHLET PROBLEM IN THE PLANE WITH RATIONAL DATA

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Abstract. We consider the Dirichlet problem for the Laplace operator with rational data on the boundary of a planar domain. Our main results include a characterization of the disk as the only domain for which all solutions are rational, and a characterization of the simply connected quadrature domains as the only ones for which all solutions are algebraic of a certain type.

1. Introduction

Let Ω be a bounded domain in the plane $\mathbb{R}^2$ and assume that $\partial\Omega$, the boundary of Ω, consists of finitely many non-intersecting Jordan curves. We shall consider the Dirichlet problem

$$\begin{cases} \Delta u = 0, & \text{in } \Omega \\ u = v & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $v \in C(\partial\Omega)$ (and $C(A)$ denotes the space of continuous functions on a topological space A). It is of course well known that this Dirichlet problem has a unique solution u in $C(\overline{\Omega})$. The case where the data function v is the restriction of a polynomial in x and y is an important special case, since, by the Stone-Weierstrass theorem and the maximum principle, any solution of (1) with continuous data can be approximated uniformly on $\overline{\Omega}$ by solutions with boundary data that are restrictions of such polynomials. Now, if Ω is a disk, or more generally the interior of an ellipse, then the latter solutions are polynomials themselves (the corresponding statement holds even in higher dimensions; see e.g. [S89]). It was conjectured in [KS92] that this property characterizes the ellipses. This conjecture was recently proved by H. Render (also in higher dimensions; see [R05]). In the present paper, we shall consider a more general situation where the data function v is the restriction to $\partial\Omega$ of a rational function $R(x, y)$ whose polar variety does not meet $\partial\Omega$.

If the boundary of Ω is real-analytic, then the solution of (1), with a restricted rational function (without poles on $\partial\Omega$) as data, extends harmonically to a larger domain. We will be interested in characterizing those domains Ω for which this harmonic extension

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encounters only "mild" singularities. For instance, if Ω is a disk, then the solution itself is rational and extends to the Riemann sphere with only a finite set of poles (see below). One of our main results is that this property characterizes the disk (with a vengeance; see Theorem 1). We also show that if all solutions are algebraic (and hence extend as multi-valued functions to the Riemann sphere minus a finite set of points at which only algebraic singularities are encountered), then Ω is simply connected and a Riemann map to the unit disk is algebraic (Theorem 4). We further characterize those simply connected domains with algebraic Riemann maps for which all solutions are algebraic with singularities that are controlled by those of the Riemann map (in a sense made precise below; see Theorem 6).

We now proceed to formulate the results of this paper more precisely. Our first result, which was alluded to above, is the following.

Theorem 1. *Let Ω be a bounded domain in $\mathbb{R}^2$ whose boundary consists of finitely many non-intersecting Jordan curves. The following are equivalent:*

(i) *Ω is a disk.*

(ii) *The solution $u(x,y)$ of (1) is rational for every $v \in C(\partial\Omega)$ that is the restriction of a rational function $R(x,y)$ whose polar variety does not meet $\partial\Omega$.*

The implication (i) $\Longrightarrow$ (ii) is easy (see e.g. [EKS05] for a proof; see also [E92]). It is appropriate to remark here that the corresponding implication (i) $\Longrightarrow$ (ii) is false in all dimensions ≥ 3 (cf. [EKS05]). The opposite implication follows from the more general result Theorem 2 below. To state it in a more convenient way, we shall identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ in the usual way, i.e. via $z = x + iy$. By the relations $2x = z + \bar{z}$ and $2iy = z - \bar{z}$, any real-analytic function $v(x,y)$ can be expressed as a function $\tilde{v}(z,\bar{z})$. We shall abuse the notation slightly and write either $v(x,y)$ and $v(z,\bar{z})$ (i.e. dropping the $\tilde{}$) for the same function v. Clearly, $v(x,y)$ is rational as a function of x and y if and only if $v(z,\bar{z})$ is rational as a function of z and $\bar{z}$.

Theorem 2. *Let Ω be a bounded domain in $\mathbb{C}$ whose boundary consists of finitely many non-intersecting Jordan curves and let $a \in \Omega$. Suppose that the solution $u(z,\bar{z})$ of (1) is rational for every $v \in C(\partial\Omega)$ that is the restriction of $R(z,\bar{z})$, where $R(z,\bar{z})$ ranges over all polynomials of z and $\bar{z}$, and the single function*

$$R(z,\bar{z}) = 1/(z-a) \tag{2}$$

Then, Ω is a disk.

Remark 3. (a) Our proof of Theorem 2 actually shows that if Ω, in addition, is assumed to be *simply connected*, then it suffices to let $R(z,\bar{z})$ range over the four functions $z\bar{z}$, $z^2\bar{z}$, $z^3\bar{z}$, and (2). The conclusion is again that Ω is a disk. (b) In this context, we should also mention a result from [EV05]: If Ω is simply connected, then the solution to (1) with data v given by the restriction of (2) is rational if and only if the solution to (1) is rational for every data v that is the restriction of a rational function of z alone. (c) Finally, we point out that the solution of (1) with data v given by the restriction of (2) is closely related to the Bergman kernel of the domain Ω (see [B95]; see also Proposition 8 below).

We shall also consider the case where the solutions $u(z,\bar z)$ to (1) with rational data (in the sense described by Theorem 1 above) are only real-algebraic; i.e. $u(z,\bar z)$ satisfies a polynomial relation $P(z,\bar z,u(z,\bar z))=0$, where $P(z,w,t)$ is a polynomial of three variables. (Note that $u(z,\bar z)$ is rational as a function of z and $\bar z$ precisely when it is real-algebraic and $P(z,w,t)$ has degree one in t. Also, note that a function $u(z,\bar z)$ is real-algebraic if and only if the polarized, or complexified, holomorphic function $u(z,\zeta)$ is algebraic.) We have the following result.

Theorem 4. *Let Ω be a bounded domain in $\mathbb{C}$ whose boundary consists of finitely many non-intersecting Jordan curves, and let $a\in\Omega$. Suppose that the solution $u(z,\bar z)$ of* (1) *is real-algebraic for every $v\in C(\partial\Omega)$ that is the restriction of $R(z,\bar z)$, where $R(z,\bar z)$ ranges over all polynomials of z and $\bar z$, and the single function* (2). *Then, Ω is simply connected and every Riemann map $\varphi\colon\Omega\to\mathbb{D}$ is algebraic.*

Now, suppose that Ω is simply connected and that a Riemann map $\varphi\colon\Omega\to\mathbb{D}$ is algebraic. We shall give a result characterizing those domains for which the solutions to (1) with rational data are real-algebraic with singularities controlled by those of the Riemann map. To explain this more precisely, we need to introduce some more notation. We let X be the (compact) Riemann surface of φ realized as a branched cover $\pi\colon X\to\mathbb{P}$, where $\mathbb{P}$ denotes the Riemann sphere (a.k.a. the extended complex plane), and Φ the meromorphic function on X obtained by lifting φ to X via the projection π. More precisely, there is a simply connected domain $\tilde\Omega\subset X$ and a meromorphic function Φ on X such that $\pi|_{\tilde\Omega}$ is a biholomorphism of $\tilde\Omega\to\Omega$ and $\Phi|_{\tilde\Omega}=\varphi\circ\pi|_{\tilde\Omega}$. The function Φ is called the lift of φ. If $f(z)$ is any function holomorphic in Ω, then there is a unique holomorphic function F, called the lift of f, on $\tilde\Omega$ such that $F=f\circ\pi$. If F extends as a meromorpic function on X, then we say that f lifts to a meromorphic function on X. This means, loosely speaking, that f is an algebraic function and its analytic continuation along curves on $\mathbb{P}$ can only encounter branch points at points where the Riemann map φ does. Moreover, the branching of f at such a point is "no more complicated" than that of φ. More precisely, the branching order of f at such a point divides that of φ.

In order to lift real-analytic functions in Ω, we introduce the conjugate Riemann surface X^* as follows: X^* equals X as a smooth manifold, but the coordinate charts on X^* are of the form $\{U_\alpha,\overline{\Psi_\alpha(\zeta)})$, where $\{U_\alpha,\Psi_\alpha(\zeta))$ are the coordinate charts on X. This is equivalent to saying that the holomorphic functions on X^* are of the form $\overline{H(\zeta)}$ where $H(\zeta)$ is holomorphic on X. We embed X as the diagonal $D:=\{(\zeta,\tau)\in X\times X^*\colon\tau=\zeta\}$ in $X\times X^*$. Observe that D is a totally real 2-dimensional submanifold of the 2-dimensional complex manifold $X\times X^*$, since $\tau\mapsto\tau$ is a an anti-holomorphic (conjugate of a holomorphic) mapping $X^*\to X$. Thus, we may think of $\tilde\Omega$ as a relatively open subset of $D\subset X\times X^*$. If $v(z,\bar z)$ is a real-analytic function in Ω, then there is a holomorphic function V in an open neighborhood of $\tilde{\Omega} \subset D$ in $X \times X^*$ such that $V(\zeta, \tau) = v(\pi(\zeta), \overline{\pi(\tau)})$. We will say that v lifts as a meromorphic function on $X \times X^*$ if V extends as a meromorphic function on $X \times X^*$. Observe that if $v(z, \bar{z})$ is a harmonic function in the simply connected domain Ω, then $v(z, \bar{z}) = f(z) + \overline{g(z)}$, where f and g are holomorphic in Ω. In this case, $V(\zeta, \tau) = F(\zeta) + \overline{G(\tau)}$, where F and G are the lifts of f and g, respectively. It follows that u lifts to $X \times X^*$ as a meromorphic function if and only if f and g lift to X as meromorphic functions. In this way we see that if u lifts as a meromorphic function on $X \times X^*$, then $u(z, \bar{z})$ is real-algebraic and the singularities of u are controlled (via f and g) by the singularities of the Riemann map $\varphi\colon \Omega \to \mathbb{D}$. We have the following result.

Theorem 5. *Let Ω be a simply connected domain in the plane with smooth boundary. Assume that a Riemann map $\varphi\colon \Omega \to \mathbb{D}$ is algebraic and let $\pi\colon X \to \mathbb{P}$ be the Riemann surface of φ realized as a branched cover. Let X^* denote the conjugate Riemann surface. The following are equivalent:*

(i) *The inverse $\varphi^{-1}\colon \mathbb{D} \to \Omega$ is rational (i.e. Ω is a quadrature domain).*

(ii) *The solution $u(z, \bar{z})$ to* (1) *lifts as a meromorphic function on $X \times X^*$ for every $v \in C(\partial\Omega)$ that is the restriction of a rational function $R(z, \bar{z})$ whose polar variety does not meet $\partial\Omega$.*

The implication (i) $\Longrightarrow$ (ii) will follow from a result in [E92] (see Section 5). The opposite implication is a consequence of the following more general result.

Theorem 6. *Let Ω be a simply connected domain in the plane with smooth boundary. Assume that a Riemann map $\varphi\colon \Omega \to \mathbb{D}$ is algebraic and let $\pi\colon X \to \mathbb{P}$ be the Riemann surface of φ realized as a branched cover. Let X^* denote the conjugate Riemann surface. If the solution $u(z, \bar{z})$ to* (1) *lifts as a meromorphic function on $X \times X^*$ for every $v \in C(\partial\Omega)$ that is the restriction of a polynomial $R(z, \bar{z})$, then the inverse $\varphi^{-1}\colon \mathbb{D} \to \Omega$ is rational (i.e. Ω is a quadrature domain).*

For our last result, we need to introduce some more notation. Suppose that $u(z, \bar{z})$ is a harmonic function in a domain $G \subset \mathbb{C}$. Let $\gamma\colon [0, 1] \to G$ be a closed piecewise smooth curve and define the *period* of $u(z, \bar{z})$ relative to γ by

$$\operatorname{per}(u; \gamma) := \int_\gamma *du, \tag{3}$$

where $*$ is the (Hodge) star operator; i.e. $*du = -u_y dx + u_x dy$. (Thus, a local harmonic conjugate of $u(z, \bar{z})$ is obtained by $v(z, \bar{z}) := \int_{z_0}^{z} *du$ for z in some small disk centered at z_0.) Observe that $*du$ is a closed 1-form and, hence, the period with respect to a curve γ only depends on the homotopy class of γ. We shall say that $u(z, \bar{z})$ is *period free* if $\operatorname{per}(u; \gamma) = 0$ for every closed piecewise smooth curve γ in G. (Thus, if u is period free in G, then u has a single-valued harmonic conjugate in G.) The last result we formulate is the following.

Theorem 7. *Let Ω be a simply connected domain in the plane with smooth boundary. Assume that a Riemann map $\varphi\colon \Omega \to \mathbb{D}$ is algebraic and that, for every $v \in C(\partial\Omega)$ that is the restriction of a polynomial $R(z, \bar{z})$, there is a discrete subset $A \subset \mathbb{C}$ (possibly depending on v) such that the solution $u(z, \bar{z})$ to* (1) *extends as a period free harmonic function in $\mathbb{C} \setminus A$. Then, Ω is a disk.*

We remark that the conclusion of Theorem 7 is not true without the assumption that the Riemann map is algebraic. For instance, as mentioned above, if Ω is an ellipse, then every solution to the Dirichlet problem (1) with polynomial data is a polynomial and, hence, extends to $\mathbb{C}$ as a period free harmonic function (see e.g. [S92]). There are also other domains Ω (with non-algebraic Riemann maps, of course) for which all solutions to the Dirichlet problem with polynomial data extend as period free harmonic functions to $\mathbb{C} \setminus A$ for some discrete set A (see [E92]).

The paper is organized as follows. In Section 2, the proof of Theorem 4 is given and, at the same time, a preliminary reduction in the proof of Theorem 2 is given as well. The proof of Theorem 2 is completed in Section 3. In Section 4, we carry out a geometric construction that will be needed for the proofs of Theorems 6 and 7. The latter proofs, as well as that of Theorem 5, are then given in Section 5.

2. Reduction to the simply connected case and proof of Theorem 4

Our first observation concerning Theorems 2 and 4 is that the boundary $\partial\Omega$ must be real-algebraic, i.e. contained in the zero locus of a (non-trivial) real polynomial. For, if the solution to (1), with v being the restriction to $\partial\Omega$ of, say, $R(z, \bar{z}) = |z|^2$ (or any other non-harmonic real-algebraic function), is real-algebraic, then $\partial\Omega$ is locally defined near each boundary point z_0 by the (non-trivial) real-algebraic equation

$$u(z, \bar{z}) - |z|^2 = 0. \tag{4}$$

It is well known that this implies that $\partial\Omega$ is in fact contained in the zero locus of a real polynomial $\rho(z, \bar{z})$. All we shall need here, however, is the fact (which actually follows immediately from (4)) that the boundary $\partial\Omega$ is piecewise real-analytic.

In order to reduce to the case where the domain Ω in Theorems 2 and 4 is simply connected, we shall need to introduce the Bergman kernel $K(z, w) = K_\Omega(z, w)$ of Ω and connect it to the solution of a particular Dirichlet problem. Recall that $K(z, w)$ is the reproducing kernel in the Bergman space $A^2(\Omega)$ of L^2-integrable holomorphic functions in Ω; i.e. $K(z, w)$ is the unique holomorphic function on $\Omega \times \Omega^*$ (where Ω^* is the domain equipped with the conjugate complex structure; or, equivalently, a function on $\Omega \times \Omega$ that is holomorphic in z and anti-holomorphic in w) such that, for every $\zeta \in \Omega$,

$$f(\zeta) = \int_\Omega f(w) K(\zeta, w) dA, \quad f \in A^2(\Omega), \tag{5}$$

where dA denotes the standard area measure in $\mathbb{R}^2$. On the other hand, if $f(z)$ is a holomorphic function in Ω that extends continuously to $\overline{\Omega}$, then we have (equipping $\partial\Omega$ with the positive orientation with respect to Ω)

$$f(\zeta) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(w)}{w-\zeta}\, dw, \tag{6}$$

Let us fix $\zeta \in \Omega$ and let $u(z,\bar z)$ be the solution to the Dirichlet problem (1) where v is the restriction to $\partial\Omega$ of $R(z,\bar z) = 1/(z-\zeta)$. Recall that $u(z,\bar z)$ is continuous in $\overline{\Omega}$ and its first order derivatives are in $L^2(\Omega)$ (cf. e.g. [Ev98], Ch. II.6). We can rewrite (6) to obtain

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi i}\int_{\partial\Omega} f(w)u(w,\bar w)dw \\ &= \frac{1}{\pi}\int_{\Omega} f(w)\frac{\partial u}{\partial \bar w}(w,\bar w)dA, \end{aligned} \tag{7}$$

where the last identity follows from Stokes' theorem and the fact that $f(w)$ is holomorphic in Ω. Observe that $\partial u/\partial\bar w$ is anti-holomorphic in Ω since

$$\frac{\partial}{\partial w}\frac{\partial u}{\partial \bar w} = \frac{1}{4}\Delta u = 0.$$

Subtracting (7) from (5), we conclude that

$$g(w) := \overline{K(\zeta,w)} - \frac{1}{\pi}\overline{\frac{\partial u}{\partial \bar w}}(w,\bar w)$$

is an element of $A^2(\Omega)$ that is orthogonal to every $f \in A^2(\Omega)$ that extends continuously to the boundary. Since the latter are dense in $A^2(\Omega)$ (see [H72]), we conclude that $K(\zeta,\cdot) = \pi^{-1}\partial u/\partial \bar w$. If we also recall that $K(z,\zeta)$ satisfies the Hermitian symmetry

$$\overline{K(\zeta,z)} = K(z,\zeta),$$

then we may write this as $K(\cdot,\zeta) = \pi^{-1}\overline{\partial u/\partial \bar w}$. We summarize this discussion as follows (see also [B92], p. 97, for the smoothly bounded case).

Proposition 8. *Let Ω be a bounded domain in $\mathbb{C}$ whose boundary consists of finitely many piecewise smooth Jordan curves, and pick $\zeta \in \Omega$. Let $u(z,\bar z)$ be the solution to (1), where v is the restriction to $\partial\Omega$ of $R(z,\bar z) := 1/(z-\zeta)$. If $K(\zeta,z)$ denotes the Bergman kernel of Ω, then*

$$K(\zeta,z) = \frac{1}{\pi}\frac{\partial u}{\partial \bar z}(z,\bar z) \tag{8}$$

or, equivalently,

$$K(z,\zeta) = \frac{1}{\pi}\overline{\frac{\partial u}{\partial \bar z}(z,\bar z)}. \tag{9}$$

We shall use this proposition to prove Theorem 4 and to reduce the proof of Theorem 2 to the case of a simply connected Ω for which a Riemann mapping $\Omega \to \mathbb{D}$ is rational. The arguments in the latter reduction are almost identical to those used to prove theorem 4. Therefore, we shall begin with the proof of Theorem 4.

Proof of Theorem 4. We first prove that Ω must be simply connected. To do this, we assume, in order to obtain a contradiction, that Ω is not. Since Ω is bounded by a finite number of Jordan curves, the complement $\mathbb{C} \setminus \overline{\Omega}$ has a finite number of bounded components $D_1, \ldots, D_n$ with $n \geq 1$. Pick a point a_j in D_j for $j = 1, \ldots, n$. It is well known that any harmonic function $u(z, \bar{z})$ in Ω can be represented as follows

$$u(z, \bar{z}) = f(z) + \overline{g(z)} + \sum_{j=1}^{n} c_j \log |z - a_j|^2, \tag{10}$$

where f and g are holomorphic in Ω and c_j are real constants. In fact, if we choose closed piecewise smooth curves $\gamma_j \colon [0, 1] \to \Omega$, for $j = 1, \ldots, n$, such that the winding number of γ_j with respect to a_l is one for $j = l$ and zero otherwise, then the constants c_j are given by

$$c_j = \frac{1}{4\pi} \mathrm{per}(u; \gamma_j). \tag{11}$$

(Recall that $\mathrm{per}(u, \gamma_j)$ denotes the period of u with respect to Ω; see (3).) Indeed, a straightforward calculation shows that the difference $\tilde{u}(z, \bar{z}) := u(z, \bar{z}) - \sum c_j \log |z - a_j|^2$ has period zero around each of the curves γ_j, $j = 1, \ldots, n$. Since these curves generate the fundamental group of Ω, we conclude that $\tilde{u}(z, \bar{z})$ is period free in Ω and is hence of the form $\tilde{v}(z, \bar{z}) = f(z) + \overline{g(z)}$ with f and g as above.

We claim that if $u(z, \bar{z})$ is a *real-algebraic* harmonic function in Ω, then necessarily the constants c_j in (10) must all be zero, i.e. $u(z, \bar{z})$ is period free in Ω. To see this, we pick a point z_l on the curve γ_l, for some $l = 1, \ldots, n$, and represent $u(z, \bar{z})$ locally near z_l as $u(z, \bar{z}) = F(z) + \overline{G(z)}$, where F and G are holomorphic near z_l. Note that both F and G are algebraic functions, since, by polarizing $u(z, \bar{z})$ to a holomorphic algebraic function $u(z, \zeta)$ and fixing, say, $\zeta = \bar{z}_l$, we conclude e.g. $F(z) = u(z, \bar{z}_l) - \overline{G(z_l)}$, which is an algebraic function of z. (Of course, a similar argument works for G). On the other hand, by choosing f and g (which are only determined up to an additive constant) and branches of the logarithm properly in (10), we must also have (near z_l)

$$F(z) = f(z) + \sum_{j=1}^{n} c_j \log(z - a_l), \quad G(z) = g(z) + \sum_{j=1}^{n} c_j \log(z - a_l). \tag{12}$$

Analytic continuations of f, g, and $\log(z - a_j)$ with $j \neq l$ around γ_l leave these functions invariant, whereas analytic continuation of $\log(z - a_l)$ around γ_l results in an additive increment of $2\pi i$. Thus, F and G would have an infinite number of branches at z_l unless the constant c_l is zero. Since F and G are algebraic, we conclude that $c_j = 0$. Since l was arbitrary in $\{1, \ldots, n\}$, we conclude that all the c_j are zero.

By the Stone-Weierstrass theorem and the maximum principle, any harmonic function $v(z, \bar{z})$ in Ω that extends continuously up to the boundary can be approximated uniformly in $\overline{\Omega}$ by solutions $u(z, \bar{z})$ to the Dirichlet problem (1) whose boundary data v are restrictions to $\partial\Omega$ of polynomials in z and $\bar{z}$. By assumption, the latter harmonic functions are real-algebraic and, hence, period free in Ω as shown above. Also, note that if a sequence u_i of harmonic functions converges uniformly in $\overline{\Omega}$ to a harmonic function u, then the derivatives of u_i converge to the corresponding derivatives of u uniformly on compact subsets of Ω. It follows, in particular, that any harmonic function in Ω that extends continuously to the boundary is period free in Ω. This is clearly a contradiction, since e.g. $\log |z - a_j|^2$ is harmonic in Ω, continuous up to the boundary, but has period 4π with respect to γ_j. Hence, Ω must be simply connected.

To complete the proof of Theorem 4, we must show that every (or, equivalently, any given) Riemann map $\varphi\colon \Omega \to \mathbb{D}$ is algebraic. Thus, we let $\varphi\colon \Omega \to \mathbb{D}$ be a Riemann map. We may assume, by composing φ with an automorphism of the disk if necessary, that $\varphi(a) = 0$. Also, let $K(z, \zeta)$ be the Bergman kernel of Ω. The Bergman kernel and the Riemann mapping are related by the following well known formula

$$K(z, \zeta) = \frac{1}{\pi} \frac{\varphi'(z)\overline{\varphi'(\zeta)}}{(1 - \varphi(z)\overline{\varphi(\zeta)})^2}. \tag{13}$$

By setting $\zeta = a$, we obtain

$$K(z, a) = \frac{\overline{\varphi'(a)}}{\pi}\varphi'(z). \tag{14}$$

Let $w(z, \bar{z})$ be the real-algebraic harmonic function $\overline{u(z, \bar{z})}$, where $u(z, \bar{z})$ solves the Dirichlet problem (1) in which v is the restriction of $1/(z - a)$. By Proposition 8, we have

$$K(z, a) = \frac{\partial w}{\partial z}(z, \bar{z}).$$

Hence, we may rewrite (14) as follows

$$\varphi'(z) = \pi(\overline{\varphi'(a)})^{-1} \frac{\partial w}{\partial z}(z, \bar{z}).$$

It follows that

$$\varphi(z) = \pi(\overline{\varphi'(a)})^{-1} w(z, \bar{z}) + \overline{h(z)},$$

where $h(z)$ is holomorphic in Ω. By polarizing, we obtain

$$\varphi(z) = \pi(\overline{\varphi'(a)})^{-1} w(z, \zeta) + \overline{h(\bar{\zeta})}.$$

Since $z \mapsto w(z, \zeta)$ is algebraic for each fixed ζ, we conclude that φ is algebraic, which completes the proof. $\square$

A trivial modification of the proof above (simply replacing "real-algebraic" by "rational") yields the following result, which reduces the proof of Theorem 2 to the situation where Ω is simply connected with a rational Riemann map.

Proposition 9. *Let Ω be a bounded domain in $\mathbb{C}$ whose boundary consists of finitely many piecewise smooth Jordan curves, and let $a \in \Omega$. Suppose that the solution $u(z, \bar{z})$ of* (1) *is rational for every $v \in C(\partial\Omega)$ that is the restriction of $R(z, \bar{z})$, where $R(z, \bar{z})$ ranges over all polynomials of z and $\bar{z}$, and the single function* (2)*. Then, Ω is simply connected and any Riemann mapping $\varphi\colon \Omega \to \mathbb{D}$ is rational.*

3. Proof of Theorem 2

In this section, we shall give a proof of Theorem 2. We first note that, by Proposition 9, we may assume that Ω is simply connected and that a Riemann mapping $\varphi\colon \Omega \to \mathbb{D}$ is rational. In particular, the boundary of Ω is piecewise real-analytic (even real-algebraic). We shall need the following lemma.

Lemma 10. *Let Ω be a bounded simply connected domain in $\mathbb{C}$. Let U be an open disk that intersects the boundary of Ω and assume that $\Gamma := \partial\Omega \cap U$ is a piecewise smooth curve without outward pointing (relative to Ω) cusps. Suppose that $u(z, \bar{z})$ is a harmonic function that extends continuously up to Γ. If f and g are holomorphic functions in Ω such that $u(z, \bar{z}) = f(z) + \overline{g(z)}$, and f and g extend meromorphically to U, then f and g cannot have poles on Γ.*

Remark 11. The conclusion of the lemma is false for outward pointing cusps in general. Consider the curve Γ defined by $y^2 - x^p = 0$ with p odd and $p \geq 5$. The harmonic function

$$u(z, \bar{z}) := \frac{1}{z} - \frac{1}{\bar{z}}$$

is harmonic in the domain $\Omega := \{y^2 - x^p < 0,\ x < 1\}$, continuous up to the boundary, but $f(z) = 1/z$ and $g(z) = -1/z$ have poles at 0.

Lemma 10 can be shown to follow from an L^2 estimate in [F37] (see also [S80] and [S81]). For completeness and the reader's convenience, we include here a self-contained proof of the lemma.

Proof of Lemma 10. Let z_0 be a point on Γ. Since z_0 is not an outward pointing cusp (by assumption), we can find an open cone V with opening angle $2\alpha > 0$ and vertex at z_0 such that $V \cap \{|z - z_0| < \delta\}$ is contained in Ω. After a translation and rotation, we may assume that $z_0 = 0$ and the cone is $V = \{z \neq 0\colon -\alpha < \mathsf{arg} z < \alpha\}$. Suppose, in order to reach a contradiction, that f and/or g have a pole at $z_0 = 0$. By simply considering the rate of growth as we approach the origin, it is easy to see (since u is continuous up to Γ) that both f and g must have a pole at 0 of the same order $p \geq 1$. We write the Laurent series expansions of f and g at 0 as follows

$$f(z) = az^{-p} + O(z^{-p+1}), \quad g(z) = bz^{-p} + O(z^{-p+1}). \tag{15}$$

Let $\gamma\colon [0,\epsilon) \to \mathbb{C}$ be a real-analytic curve such that $\gamma(t)$ is of the form

$$\gamma(t) = \zeta t + O(t^2), \tag{16}$$

where $\zeta \neq 0$ is in the cone V. Clearly, γ (or more precisely, the image of γ) is a curve starting at the origin and, for $\epsilon > 0$ small enough, $\gamma \setminus \{0\}$ is contained in Ω. We consider the restriction of $u(z,\bar{z})$ to γ

$$\begin{aligned} u(\gamma(t),\overline{\gamma(t)}) =& f(\gamma(t)) + \overline{g(\gamma(t))} \\ =& a(\zeta t)^{-p} + \overline{b(\zeta t)^{-p}} + O(t^{-p+1}) \\ =& \frac{a\bar{\zeta}^p + \bar{b}\zeta^p}{|\zeta|^{2p}} t^{-p} + O(t^{-p+1}), \end{aligned} \tag{17}$$

where in the second step we used the Laurent expansions in (15). By assumption, $u(\gamma(t),\overline{\gamma(t)})$ has a limit as $t \to 0^+$ and, hence, we must have (at the very least)

$$\frac{a\bar{\zeta}^p + \bar{b}\zeta^p}{|\zeta|^{2p}} = 0, \tag{18}$$

or equivalently,

$$a\bar{\zeta}^p + \bar{b}\zeta^p = 0, \tag{19}$$

This equation only has non-trivial solutions if $|a| = |b|$ and then $\zeta^p = x + iy$ is on the line

$$(\mathrm{Re}\ a + \mathrm{Re}\ b)x + (\mathrm{Im}\ a + \mathrm{Im}\ b)y = 0. \tag{20}$$

This is a contradiction, since ζ is an arbitary point in the open cone V. This proves that f and g cannot have poles at $z_0 \in \Gamma$ and hence, since z_0 is arbitrary, at any point on Γ. $\square$

We now return to the proof of Theorem 2. Recall that we have reduced the proof to the case where Ω is simply connected, the boundary is piecewise real-analytic, and a Riemann map $\varphi\colon \Omega \to \mathbb{D}$ is rational. We claim that we can further assume that if $u(z,\bar{z})$ is the (rational) solution to one of the Dirichlet problems in the statement of Theorem 2 and $u(z,\bar{z}) = f(z) + \overline{g(z)}$, with f and g holomorphic in Ω, then f and g are both rational functions without poles on $\overline{\Omega}$. The fact that f and g are rational is clear by polarizing $u(z,\bar{z})$ to a rational function $u(z,\zeta) = f(z) + \bar{g}(\zeta)$, where $\bar{g}(\zeta) := \overline{g(\bar{\zeta})}$, and using the fact that $z \to u(z,\zeta)$ and $\zeta \to u(z,\zeta)$ are rational for any fixed ζ and z, respectively (cf. the proof of Theorem 4). The fact that f and g have no poles on $\partial\Omega$ (and hence not in $\overline{\Omega}$) follows from Lemma 10 if we can show that $\partial\Omega$ does not have any outward pointing cusps. This is not difficult to see. Since the Riemann map φ is rational and bounded in Ω, it must be holomorphic in an open neighborhood of $\overline{\Omega}$, sending $\partial\Omega$ one-to-one (since $\partial\Omega$ is, in particular, a Jordan curve) onto $\partial\mathbb{D}$. Under these conditions, it is easy to see that $\partial\Omega$ cannot have any cusps (outward or inward pointing). We leave the details of this argument to the reader.

For our purposes, it will be convenient to compose the Riemann mapping $\varphi\colon \Omega \to \mathbb{D}$ with a Möbius transformation $\mathbb{D} \to \mathbb{U}$, where $\mathbb{U}$ denotes the upper half plane $\{z = x+iy\colon y>0\}$. We shall denote the resulting rational biholomorphic mapping $\Omega \to \mathbb{U}$ by ψ. The Riemann sphere (a.k.a. extended complex plane) $\mathbb{P}$ is subdivided into regions $G_1, G_2, \ldots, G_N$ by the piecewise real analytic curves that comprise $\psi^{-1}(\mathbb{R})$. Let $G_1 = \Omega$. We shall describe a procedure that creates an increasing union of the closures of these domains that covers the whole Riemann sphere. Later, as we piece together the regions, we will show that ψ must be linear fractional, and hence, that Ω must be a disc.

Notice that ψ is a proper holomorphic mapping of each domain G_j onto either the upper or the lower half plane, and as such, is a finite-to-one branched covering map between the two domains. When we say that we are choosing a branch of ψ^{-1} below, we mean that we are choosing either the upper half plane or the lower half plane and we are thinking of ψ^{-1} as the continuation of a local inverse of ψ, where ψ is viewed as a proper holomorphic mapping of one of the G_j's onto the half plane. Note that we may continue any branch of ψ^{-1} as a finite valued holomorphic function with only finitely many algebraic singularities in the half plane.

Let $S(z)$ denote the Schwarz function for a smooth part of the boundary of Ω near a point $z_0 \in \partial\Omega$; i.e. $S(z)$ is the holomorphic function near z_0 such that $S(z) = \bar{z}$ on $\partial\Omega$. The anti-holomorphic *Schwarz reflection function* for the boundary of Ω near z_0 is given the by

$$\overline{S(z)} = \psi^{-1}(\overline{\psi(z)}),$$

where ψ^{-1} is holomorphic near $\psi(z_0) \in \mathbb{R}$ and is the inverse to ψ viewed as a one-to-one map on a neighborhood of z_0. This mapping, defined near a point z_0 in a smooth part of the boundary of Ω, analytically continues to all of $\mathbb{P}$ as an anti-holomorphic (multi-valued) algebraic function.

When we continue $\overline{S(z)}$ to various of the G_j's, we may view it as a proper anti-holomorphic correspondence between them. Indeed, on G_j, the mapping ψ maps properly to either the upper or the lower half plane. The conjugate map reflects $\psi(z)$ across the real line to $\overline{\psi(z)}$, sending one half plane to the other, and so a branch of ψ^{-1} yields a mapping $\overline{S(z)} = \psi^{-1}(\overline{\psi(z)})$ which maps G_j onto another G_k as a finite valued anti-holomorphic function with only algebraic singularities. (The map might split into several separate irreducible correspondences, or it might yield a single irreducible correspondence. It won't matter to us what the case may be.) We may remove finitely many points from the domains G_j so that all the mappings obtained in this way are unbranched covering maps. Indeed, let $\mathcal{A}$ denote the set of finitely many branch points of ψ in the union of all the G_j. Given a set U of complex numbers, let $\mathrm{conj}(U) = \{\bar{w} : w \in U\}$ denote the set of conjugates. Let $\mathcal{F} = \mathcal{A} \cup \psi^{-1}(\mathrm{conj}(\psi(\mathcal{A}))$. Replace each G_j by $G_j - \mathcal{F}$. In this way, the various branches of $\overline{S(z)}$ become finitely sheeted covering maps of one G_j onto another. Notice that if $\overline{S(z)}$ maps G_j onto G_k, then as z approaches a point in the boundary of

G_j along a curve in G_j, the analytic continuation of $\overline{S(z)}$ along that curve tends to a boundary point of G_k.

Pick a point z_0 in the smooth part of the boundary of Ω and consider a curve γ parametrized by $z(t)$ that starts at z_0 and heads into the exterior of Ω. Let Γ denote the "shadow curve" of γ parametrized by $Z(t) = \overline{S(z(t))}$, where it is understood that $Z(t)$ is produced by analytically continuing $\overline{S(z)} = \psi^{-1}(\overline{\psi(z)})$ along the trace of γ, i.e., by analytically continuing ψ^{-1} along the curve traced out by $\overline{\psi(z(t))}$. Since we have arranged for all of our maps to be covering maps, there are no obstructions to this continuation process. As $z(t)$ enters the exterior of Ω, the function $\psi(z(t))$ enters the lower half plane, the reflected $\overline{\psi(z(t))}$ enters the upper half plane, and $Z(t) = \psi^{-1}(\overline{\psi(z(t))})$ enters the interior of Ω. Recall that G_1 is equal to Ω minus perhaps finitely many points. Let G_{2j}, $j = 1, \ldots, M_2$, be an enumeration of the regions in the list that share an open smooth segment of a boundary curve in common with G_1. We shall call G_1 our level 1 region and the G_{2j} our level 2 regions. Note that there could be only one level 2 region if the boundary of Ω is a smooth curve, or there could be more than one if the boundary of Ω has a number of corners where the real analytic curves in $\psi^{-1}(\mathbb{R})$ cross. Call the closure of G_1 our *stage 1* set, and the closure of $G_1 \cup (\cup_{j=1}^{M_2} G_{2j})$ our *stage 2* set. Notice that for any curve γ that starts at z_0 and wanders about a G_{2j}, the shadow curve Γ wanders about G_1. We can express this by saying that as γ wanders around a level 2 region, the shadow curve wanders in our level 1 region. Furthermore, if γ is a curve in G_{2j} that terminates at a boundary point of G_{2j}, then the shadow curve is a curve in G_1 that terminates at a boundary point of G_1. Thus, as γ is allowed to wander about the level 2 region and terminate at points in its boundary, the shadow curve stays in the stage 1 set.

We next define our level 3 regions to be those regions which share an open smooth segment of a boundary curve with a level 2 region, but not with G_1 (if such regions exist). Let G_{3m}, $m = 1, \ldots, M_3$, be an enumeration of such regions. Now let γ be a curve that starts at z_0 and moves through a region G_{2j} and comes to a point on an open smooth segment of a curve in $\psi^{-1}(\mathbb{R})$ that connects G_{2j} to G_{3m}. As $z(t)$ extends into G_{3m}, the point $Z(t) = \psi^{-1}(\overline{\psi(z(t))})$ in the shadow curve Γ leaves G_1 and enters a region G_{2k}. Now, as γ continues to wander around G_{3m}, the shadow curve wanders around G_{2k} because the branch of $\psi^{-1}(\overline{\psi(z)})$ that we continue into G_{3m} is an unbranched anti-holomorphic covering map of G_{3m} onto G_{2k}. If γ is allowed to run into a boundary point of G_{3m}, then the shadow curve runs into a boundary point of G_{2k}. Thus, as γ is allowed to wander about the level 3 region and terminate at points in its boundary, the shadow curve stays in the stage 2 set.

Call the closure of the union of G_1 and all the G_{2k} regions and all the G_{3m} regions the stage 3 set.

The process above continues in an obvious way, however, at the next level, level 4, as we allow γ to cross a level 3 boundary curve and enter a level 4 domain G_{4j} (which is a domain which shares a boundary curve with a level 3 domain, but no earlier level domain), it may happen that the shadow curve goes back into an earlier level domain than level 3. That will cause us no problems. We care only that the shadow curve is always *at least* one level behind the curve we extend. We may always state that the shadow curve stays in the stage $(n-1)$ *set* as γ wanders about the level n domain and terminates at points on its closure.

At some stage, when we add the closure of a level N domain to the stage $N-1$ set, we will cover the entire Riemann sphere. We shall call this level N the *last level.* (Note, it could happen that $N = 2$.) We shall need it to happen that the point at infinity falls in the interior of the last level domain. To ensure that this is the case, we may modify our original domain Ω using a linear fractional transformation. Indeed, if the point at infinity is not in the interior of the last level domain, then pick any point p_0 that is. Let $L(z) = 1/(z-p_0)$, and replace our original domain by $L(\Omega)$. This new domain still satisfies the hypothesis of the theorem because linear fractional transformations and their inverses preserve rational functions. Furthermore, the sequence of levels and stages that we constructed above is simply picked up and moved by L on the Riemann sphere.

We may now begin the crux of the proof. The boundary data $\bar{z}z^n$ has a harmonic extension to Ω given by $u_n(z,\bar{z}) = f_n(z) + \overline{g_n(z)}$ where f_n and g_n are rational functions of z with no poles on $\overline{\Omega}$ (by Lemma 10, as observed above). The Schwarz function is holomorphic on a neighborhood of $\partial\Omega$ and satisfies $S(z) = \bar{z}$ on $\partial\Omega$. We may insert this fact into the identity

$$\bar{z}z^n = f_n(z) + \overline{g_n(z)}$$

and its conjugate

$$z\bar{z}^n = \overline{f_n(z)} + g_n(z),$$

which hold on $\partial\Omega$, to obtain the identities

$$\begin{aligned} S(z)z^n &= f_n(z) + \overline{g_n(\overline{S(z)})}, \text{ and} \\ zS(z)^n &= \overline{f_n(\overline{S(z)})} + g_n(z), \end{aligned}$$

which hold for $z \in \partial\Omega$. Since the functions on both sides of these identities are holomorphic on a neighborhood of the point $z_0 \in \partial\Omega$ from which we started the curve γ, the identities extend to hold on this neighborhood, and they analytically continue to hold as we continue $S(z)$ along any curve. (Note that $\overline{S(z)}$ might develop bounded algebraic singularities at finitely many points that fall in the boundaries of the various G_j's, but these points will pose no threat to us as boundary points.) Rewrite the last two formulas to read

$$f_n(z) = S(z)z^n - \overline{g_n(\overline{S(z)})} \tag{21}$$

$$g_n(z) = zS(z)^n - \overline{f_n(\overline{S(z)})}. \tag{22}$$

Recall that as the curve γ parametrized by $z(t)$ moved into G_2, the shadow curve Γ given by $Z(t) = \overline{S(z(t))}$ moved into G_1. Since f_n and g_n have no poles in the closure of G_1

(which is equal to the closure of Ω), equations (21) and (22) reveal that f_n and g_n have no poles in the closure of G_2 either. (Note that $S(z)$ remains bounded because it is the conjugate of $\overline{S(z)}$, which stays in the bounded domain G_1.)

This process may be continued as we repeat the process of extending the curve γ and following its shadow Γ as we did in the construction of the levels and the stages. Since $\overline{S(z(t))}$ is always at least one level behind $z(t)$ as we analytically continue $\overline{S}$ along $z(t)$, we show level by level that f_n and g_n are pole free. This procedure goes smoothly until we reach the last level and wonder what the point at infinity holds in store for us. As we let γ run out to infinity in the last level domain via a parametrization $z(t)$, the shadow Γ tends to a point in the finite complex plane in the closure of a previous level. Equation (21) shows that f_n has at worst a pole of order n at infinity and equation (22) shows that g_n has at worst a pole of order 1 at infinity. Since these are the only poles, we conclude that f_n is a polynomial of degree at most n and $g_n(z) = Az + B$ for some constants A and B.

Now, if any function g_n were to be the zero function, then identity (21) would yield that $S(z)$ is a rational function, and it would follow from a theorem of Davis [D74] that Ω would have to be a disc, and the conclusion of the theorem holds true. So we need only consider the case where g_1, g_2, and g_3 are non-zero. Any three non-zero first degree polynomials are linearly dependent, and so there exist constants, c_1, c_2, c_3, not all zero, such that

$$c_1\overline{g_1(z)} + c_2\overline{g_2(z)} + c_3\overline{g_3(z)} \equiv 0.$$

Now, taking this same linear combination of the identities (21), we obtain

$$c_1 f_1(z) + c_2 f_2(z) + c_3 f_3(z) = S(z)(c_1 z + c_2 z^2 + c_3 z^3),$$

and we again see that $S(z)$ is rational, and Davis' theorem yields that Ω must be a disc. The proof of Theorem 2 is complete.

4. A geometric construction for certain domains with algebraic Riemann maps

In order to prove Theorems 6 and 7, we need a geometric result asserting the existence of special finite subsets in the complexified boundaries of certain simply connected domains in the plane. The inspiration for doing so comes from a paper by Hansen and Shapiro (see [HS94]) in which the existence of "rectangles" in the complexified boundaries was used to infer that certain Dirichlet problems failed to have solutions that extend real-analytically to the whole plane. To state and prove the result we need, we must introduce some notation. We shall say that a finite subset $S = \{(z_j, w_j) \in \mathbb{C}^2 \colon j = 1, \dots m\}$ of $\mathbb{C}^2$ is a *2-set* if each of the intersections $S \cap \{(z, w) \colon z = z_j\}$ and $S \cap \{(z, w) \colon w = w_j\}$, for $j = 1, \dots, n$, consists of precisely two points. Clearly, the number of points in a 2-set is always even. The simplest example example of a 2-set is the set of vertices in a rectangle $\{(a, c), (b, c), (b, d), (a, d)\}$ with $a \neq b$ and $c \neq d$. If a 2-set S is contained in a complex algebraic variety V and V is the zero locus of the polynomial $P(z, w)$, then the degree of

P as a polynomial in z (with coefficients that are polynomials in w) as well as that of P as a polynomial in w is at least 2.

Now, let Ω be a simply connected bounded domain in $\mathbb{R}^2 \cong \mathbb{C}$ with smooth real algebraic boundary, i.e. $\partial\Omega$ is a smooth curve that is contained in the zero locus of a real-valued polynomial $P(z, \bar{z})$. Without loss of generality, we may assume that $P(z, w)$ is irreducible (over $\mathbb{C}$). We shall denote by $V_{\partial\Omega}$ the complex algebraic irreducible curve $\{(z, w) \in \mathbb{C}^2 \colon P(z, w) = 0\}$. If we identify $\mathbb{R}^2 \cong \mathbb{C}$ with the anti-diagonal $\{(z, w) \colon w = \bar{z}\}$ of $\mathbb{C}^2$ in the usual way, then $V_{\partial\Omega}$ is the unique irreducible variety in $\mathbb{C}^2$ whose intersection with $\mathbb{R}^2 \cong \mathbb{C}$ contains the boundary $\partial\Omega$. We shall refer to $V_{\partial\Omega}$ as the *complexification* of $\partial\Omega$. (We should point out here that we could also have considered the complexification in the product $X \times X^*$, where $X = \mathbb{C}$ and X^* is the conjugate Riemann surface, and embedded $X = \mathbb{C}$ along the diagonal in $X \times X^*$ as in the introduction. However, in the case of $X = \mathbb{C}$, it seems more standard to consider $X \times X$ and embed $X = \mathbb{C}$ as the anti-diagonal.) For instance, the complexification of the unit circle $\{|z|^2 = 1\}$ is the variety $V = \{(z, w) \colon zw - 1 = 0\}$. Since the degree of $zw - 1$ as a polynomial in z (and also that as a polynomial in w is less than 2, it cannot contain any 2-sets as observed above. A less trivial example of a simply connected domain whose complexified boundary does not contain any 2-sets is the ellipse

$$E = \{z \colon z\bar{z} + A(z^2 + \bar{z}^2) < R\},$$

where $A \in \mathbb{R} \setminus \{0\}$ ($A = 0$ corresponds to a circle) and $R > 0$. We shall leave the verification of this to the reader. (It also follows from the arguments in [E92].) Observe that no Riemann mapping $\varphi \colon E \to \mathbb{D}$, where $\mathbb{D}$ denotes the unit disk, is algebraic. (See e.g. [N52]). The main result in this section is the following.

Theorem 12. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ with smooth boundary. Assume that a Riemann mapping $\varphi \colon \Omega \to \mathbb{D}$ is algebraic and that the inverse $\varphi^{-1} \colon \mathbb{D} \to \Omega$ is not rational. Then, the complexified boundary $V_{\partial\Omega}$ contains a family of 2-sets $S(t) = \{(z_j(t), w_j(t)) \colon j = 1, \dots n\}$, $t \in (-\epsilon, \epsilon)$, such that each set $z_j((-\epsilon, \epsilon))$ and $w_j((-\epsilon, \epsilon))$, for $j = 1, \dots n$, is a real-analytic variety of dimension one.*

Remark 13. The property that the inverse Riemann mapping $\varphi^{-1} \colon \mathbb{D} \to \Omega$ is not rational is equivalent to Ω not being a quadrature domain (see e.g. [S92]).

As an immediate corollary of Theorem 12, we obtain the following.

Corollary 14. *Let Ω be a bounded simply connected domain in $\mathbb{C}$ with smooth boundary. Assume that a Riemann mapping $\varphi \colon \Omega \to \mathbb{D}$ is rational with* $\deg \varphi \geq 2$. *Then, the complexified boundary $V_{\partial\Omega}$ contains a family of 2-sets $S(t) = \{(z_j(t), w_j(t)) \colon j = 1, \dots n\}$, $t \in (-\epsilon, \epsilon)$, such that each set $z_j((-\epsilon, \epsilon))$ and $w_j((-\epsilon, \epsilon))$, for $j = 1, \dots n$, is a real-analytic variety of dimension one.*

Proof of Theorem 12. First, it will be convenient, as in the proof of Theorem 2, to compose the Riemann mapping $\varphi \colon \Omega \to \mathbb{D}$ with a linear fractional transformation $\mathbb{D} \to \mathbb{U}$, where

$\mathbb{U}$ denotes the upper half plane. We shall denote by ψ the resulting biholomorphism $\psi\colon \Omega \to \mathbb{U}$. Observe that ψ is also algebraic. As above, we denote by $\mathbb{P}$ the Riemann sphere (a.k.a. the extended complex plane) and embed Ω, $\mathbb{U}$ in $\mathbb{P}$ via the inclusion $\mathbb{C} \subset \mathbb{P}$. Let X be the (compact) Riemann surface of ψ realized as a branched covering $\pi\colon X \to \mathbb{P}$ and Ψ the meromorphic function on X obtained by lifting ψ to X via the projection π. Recall from the introduction that there is a simply connected domain $\tilde{\Omega} \subset X$ and a meromorphic function Ψ on X such that $\pi|_{\tilde{\Omega}}$ is a biholomorphism of $\tilde{\Omega} \to \Omega$ and $\Psi|_{\tilde{\Omega}} = \psi \circ \pi|_{\tilde{\Omega}}$. Since ψ is holomorphic in a neighborhood of the closure $\overline{\Omega}$ (by the smoothness assumption of $\partial\Omega$), π is a actually a biholomorphism of a neighborhood of $\overline{\tilde{\Omega}}$ to a neighborhood of $\overline{\Omega}$. In particular, $\partial\tilde{\Omega}$ is also a smooth real-analytic curve.

We may view Ψ as a holomorphic mapping $\Psi\colon X \to \mathbb{P}$ sending $\tilde{\Omega}$ biholomorphically onto the upper half plane $\mathbb{U} \subset \mathbb{P}$. We shall let X_0 denote the open Riemann surface $X \setminus (\Psi^{-1}(\infty) \cup \pi^{-1}(\infty))$. We shall need the complexification of the (connected) real-analytic curve $\partial\tilde{\Omega} \cap X_0$. Since there is no notion of complex conjugation on X_0, we need to define the complexification of a real-analytic curve in X_0 slightly differently than we did in the case of a real-analytic curve in $\mathbb{C}$ (see above). However, it is easily seen that if $X_0 = \mathbb{C}$, then the two notions of complexification are equivalent. We denote, as in the introduction, by X_0^* the Riemann surface X_0 with the conjugate complex structure, i.e. $X_0^* = X_0$ as smooth manifolds, but the holomorphic functions on X_0^* are of the form $\overline{H(\zeta)}$ where $H(\zeta)$ is holomorphic on X_0. We embed X_0 as the diagonal $D := \{(\zeta,\tau) \in X_0 \times X_0^*\colon \tau = \zeta\}$ in $X_0 \times X_0^*$. The complexification $V_{\partial\tilde{\Omega}}$ is the unique irreducible complex analytic subvariety of $X_0 \times X_0^*$ that contains $\partial\tilde{\Omega} \subset D \subset X_0 \times X_0^*$. Since π is a biholomorphism of a neighborhood of $\overline{\tilde{\Omega}}$ onto a neighborhood of $\overline{\Omega}$, the complexification $V_{\partial\tilde{\Omega}}$ can be described as the component of the complex variety

$$\big\{(\zeta,\tau)\colon P(\pi(\zeta), \overline{\pi(\tau)}) = 0\big\}$$

that contains $\partial\tilde{\Omega} \subset D$; here, $P(z,\bar{z})$ is an irreducible polynomial whose zero locus contains $\partial\Omega$. Note that the projection

$$\Pi(\zeta,\tau) := (\pi(\zeta), \overline{\pi(\tau)}) \tag{23}$$

is a proper holomorphic mapping of $V_{\partial\tilde{\Omega}}$ to $V_{\partial\Omega}$. The notion of a 2-set generalizes easily to subsets of $X_0 \times X_0^*$: A finite subset $S = \{(\zeta_j,\tau_j) \in X_0 \times X_0^*\colon j = 1,\ldots n\}$ is a 2-set if each of the intersections $S \cap \{(\zeta,\tau)\colon \zeta = \zeta_j\}$ and $S \cap \{(\zeta,\tau)\colon \tau = \tau_j\}$, for $j = 1,\ldots,n$, consists of precisely two points. To find 2-sets on $V_{\partial\Omega}$, we shall look for 2-sets on $V_{\partial\tilde{\Omega}}$ and project these to $V_{\partial\Omega}$ by Π. More precisely, we shall need the following result.

Proposition 15. *Let $\tilde{\Omega}$, X, and Ψ be as above. Then, the complexification $V_{\partial\tilde{\Omega}} \subset X_0 \times X_0^*$ contains a family of 2-sets $S(t) = \{(\zeta_j(t),\tau_j(t))\colon j = 1,\ldots n\}$, $t \in (-\epsilon,\epsilon)$, such that each set $\zeta_j((-\epsilon,\epsilon))$ and $\tau_j((-\epsilon,\epsilon))$, for $j = 1,\ldots n$, is a real-analytic variety of dimension one.*

Moreover, there is a real-analytic mapping $x\colon (-\epsilon,\epsilon)\to\mathbb{R}$ *such that, for each* $t\in(-\epsilon,\epsilon)$, *we have* $x(t)=\Psi(\zeta_j(t))=\Psi(\tau_j(t))$ *for* $j=1,\dots,m$.

Proof of Proposition 15. We first claim that the degree of the mapping $\Psi\colon X\to\mathbb{P}$ is at least two. If it were one, then Ψ would be a biholomorphism $X\to\mathbb{P}$ (and so $X=\mathbb{P}$). Hence, $\pi\circ\Psi^{-1}$, restricted to $\mathbb{U}$ would be the inverse of ψ. However, $\pi\circ\Psi^{-1}$ would be a holomorphic mapping $\mathbb{P}\to\mathbb{P}$, i.e. a rational function, and this would contradict the assumption in Theorem 12 that φ^{-1} is not rational (since φ and ψ differ by a linear fractional transformation).

Now, let $\mathbb{L}\subset\mathbb{P}$ denote the lower half plane and $G\subset X$ be the connected component of $\Psi^{-1}(\mathbb{L})$ whose boundary meets that of $\tilde\Omega$. Since Ψ is a biholomorphic map $\tilde\Omega\to\mathbb{U}$ that extends biholomorphically across $\partial\tilde\Omega$, it follows that G is uniquely determined (and that $\partial\tilde\Omega$ is a connected component of ∂G). Since the degree of Ψ is at least two and Ψ is biholomorphic $\tilde\Omega\to\mathbb{U}$, $\Psi^{-1}(\mathbb{U})$ must have at least two connected components. Since one of these components is the simply connected domain $\tilde\Omega$, it follows that ∂G is disconnected. Now, let a be a point of $\partial\tilde\Omega\subset\partial G$ such that $\Psi(a)$ is not the point at infinity and not a critical value of Ψ; recall that the latter is equivalent to $\Psi^{-1}(\Psi(a))$ consisting of $m:=\deg\Psi$ distinct points and Ψ being a local biholomorhism near each of these points. We shall denote the points of $\Psi^{-1}(\Psi(a))$ by $a_1=a,a_2,\dots,a_m$. Since ∂G has at least two boundary components, at least one of the points $a_2,\dots,a_m$, say a_2, must be on $\partial G\setminus\partial\tilde\Omega$. Let $c\colon[0,1]\to\bar G$, with $c\colon(0,1)\to G$, be a curve starting at $a_1\in\partial\tilde\Omega\subset\partial G$ and ending at $a_2\in\partial G\setminus\partial\tilde\Omega$. We define $L\colon[0,1]\to\bar{\mathbb{L}}$, with $L\colon(0,1))\to\mathbb{L}$, by $L(t)=\Psi(c(t))$. Thus, L is a closed loop in $\mathbb{L}$ starting and ending at $x:=\Psi(a)$. Let $L^*(t)=\overline{L(t)}$. We shall require, as we may, that both L and L^* avoid the (finite number of) critical values of Ψ. Each choice of a point $a_j\in\Psi^{-1}(x)$ identifies uniquely a local inverse of Ψ near x with the property that the inverse of x is a_j. Let us label the local inverses $\Psi_1^{-1},\dots,\Psi_m^{-1}$. We obtain two permutations M and M^* on the finite set of m elements $\{1,\dots,m\}$ by the standard monodromy action of the loops L and L^* on the local inverses of Ψ. More precisely, for $j\in\{1,\dots,m\}$, we define $M(j)$ as follows. We continue Ψ_j^{-1} analytically around the loop L. Upon returning to x, the analytic continuation of Ψ_j^{-1} will still be a local inverse of Ψ. We define $M(j)$ to be the index i of the resulting local inverse. The permutation M^* is defined in the same way by performing analytic continuation along the loop L^* instead L. Observe that, by construction, $M(1)=2$. Moreover, since Ψ is a biholomorphism of $\tilde\Omega$ onto $\mathbb{U}$ sending a_1 to x, the local inverse Ψ_1^{-1} is holomorphic in the whole upper half plane and, hence, $M^*(1)=1$.

We shall use the permutations M and M^* to find 2-sets on $V_{\partial\tilde\Omega}$ as follows. We observe, since Ψ maps $\partial\tilde\Omega$ to the real line, that $V_{\partial\tilde\Omega}$ is the component of the complex variety

$$W:=\left\{(\zeta,\tau)\in X_0\times X_0^*\colon \Psi(\zeta)-\overline{\Psi(\tau)}=0\right\} \tag{24}$$

that contains $\partial\tilde{\Omega} \subset D$. Clearly, the points (a_i, a_j), for $i, j = 1, \ldots, m$, are all on the variety W (since $x = \Psi(a_i) = \Psi(a_j)$ is real) and the point (a_1, a_1) is on $V_{\partial\tilde{\Omega}}$ (since $a_1 = a \in \partial\tilde{\Omega}$). If we pick a point (a_i, a_j), continue Ψ_i^{-1} around L and Ψ_j^{-1} around L^*, and denote by $c_i(t) := \Psi_i^{-1}(L(t))$, $c_j^*(t) := \Psi_j^{-1}(L^*(t))$, then $t \mapsto (c_i(t), c_j^*(t))$ is a curve on the regular part of W connecting (a_i, a_j) to $(a_{M(i)}, a_{M^*(j)})$. Consequently, if (a_i, a_j) is on the complexification $V_{\partial\tilde{\Omega}}$, then so is $(a_{M(i)}, a_{M^*(j)})$. Completely analogous arguments show that if (a_i, a_j) is on $V_{\partial\tilde{\Omega}}$, then so are $(a_{M^*(i)}, a_{M(j)})$, $(a_{M^{-1}(i)}, a_{(M^*)^{-1}(j)})$, and $(a_{(M^*)^{-1}(i)}, a_{M^{-1}(j)})$. We now define two permutations K and K^* on the product $\{1, \ldots, m\} \times \{1, \ldots, m\}$ by

$$K(i,j) := (M(i), M^*(j)), \quad K^*(i,j) = (M^*(i), M(j)). \tag{25}$$

Let H be the group of permutations generated by K and K^*. We may summarize the discussion above in terms of the orbit of $(1,1)$ under H as follows.

Proposition 16. *Let $O \subset \{1, \ldots, m\} \times \{1, \ldots, m\}$ be the orbit of $(1,1)$ under the group H, i.e.*

$$O := \{(i,j) \colon (i,j) = h(1,1) \text{ for some } h \in H\}. \tag{26}$$

Then the finite subset

$$\tilde{S} := \{(a_i, a_j) \colon (i,j) \in O\} \tag{27}$$

is contained in $V_{\partial\tilde{\Omega}}$.

We shall now show that $\tilde{S} \subset V_{\partial\tilde{\Omega}}$ contains a 2-set S. This is equivalent to showing that O contains a subset O' such that, for every $(i,j) \in O'$, the two intersections $O' \cap \{(k,l) \colon k = i\}$ and $O' \cap \{(k,l) \colon l = j\}$ both contain precisely two points. Before doing this, however, we first claim that we may assume, without loss of generality, that the permutation group J generated by M and M^* is transitive on $\{1, \ldots, m\}$. If it is not, then it is transitive on some subset $\{1, \ldots, m'\}$ (after possibly renumbering the points $a_3, \ldots, a_m$), where $2 \leq m' \leq m$. The orbit of $(1,1)$ under H is then a subset of the product $\{1, \ldots, m'\} \times \{1, \ldots, m'\}$. The arguments below would then go through with m replaced by m' and a 2-set on $V_{\partial\tilde{\Omega}}$ would still result. Thus, we shall proceed under the assumption that J is transitive on $\{1, \ldots, m\}$.

Proposition 17. *Let O be as in Proposition 16. Then, O is symmetric, i.e. invariant under the involution $(i,j) \mapsto (j,i)$, and the intersection with each horizontal line $\{(i,j) \colon j = j_0\}$, for $j_0 = 1, \ldots, m$, contains at least two points.*

Remark 18. Since O is symmetric, the intersection with each vertical line $\{(i,j) \colon i = i_0\}$, for $i_0 = 1, \ldots, m$, contains at least two points as well.

Proof of Proposition 17. The symmetry is obvious since O is the orbit of a group. Let J, as above, be the transitive permutation group on $\{1, ..., m\}$ generated by M and M^*. We define an involution $T : J \to J$ (i.e. T is a homomorphism with $T^2 = I$) by $T(M) = M^*$.

Note that any element h in H is of the form $h(i,j) = (Tg(i), g(j))$ for some $g \in J$. Fix j_0 as in the statement of the proposition. Thus, we need to show that there are $i_1 \neq i_2$ and $g_1, g_2 \in J$ such that $g_1(1) = g_2(1) = j_0$ and $Tg_1(1) = i_1$, $Tg_2(1) = i_2$. Pick any $g_1 \in J$ such that $g_1(1) = j_0$ (which can be done by the transitivity) and set $i_1 = Tg_1(1)$. Let J_1 be the subgroup of J consisting of the elements that fix 1. The subgroup J_1 is nontrivial since M^* is in J_1, and TJ_1 is not contained in J_1 since M is not in J_1. Pick g in J_1 such that Tg is not in J_1 and set $g_2 := g_1 g$. Clearly, $g_2(1) = j_0$. Also, $Tg_2(1) = Tg_1(Tg(1)) = Tg_1(i)$, where $i := Tg(1)$ is not 1 since Tg is not in J_1. It follows that $i_2 = Tg_1(i)$ is different from $i_1 = Tg_1(1)$ since $i \neq 1$). This completes the proof of the proposition. □

We shall now construct a subset $O' \subset O$, where O is as in Proposition 16, such that, for every $(i,j) \in O'$, the two intersections $O' \cap \{(k,l) \colon k = i\}$ and $O' \cap \{(k,l) \colon l = j\}$ both contain precisely two points, which, as we recall, is equivalent to $S := \{(a_i, a_j) \colon (i,j) \in O'\}$ being a 2-set contained $V_{\partial\tilde{\Omega}}$. To this end, we consider the following construction. Set $p_0 = (1,1)$. Pick $i_1 \neq 1$ such that $p_1 = (i_1, 1)$ is in O (which is possible by Proposition 17)). By symmetry (Proposition 17)), $q_1 = (1, i_1)$ is also in O. Pick $i_2 \neq 1$ such that $p_2 = (i_2, i_1)$ is in O. If $i_2 = i_1$, then $O' := \{p_0, p_1, q_1, p_2\}$ is a 2-set and the construction is finished. If not, then we set $q_2 = (i_1, i_2)$. Observe that q_2 is also in O and $q_2 \neq p_2$. Set $i_0 := 1$. Assume that we have defined a sequence $i_0, i_1, \ldots, i_k$ such that $i_k \notin \{i_0, i_i, \ldots, i_{k-1}\}$ and such that the points $p_l := (i_l, i_{l-1})$, $l = 1, \ldots, k$, and $q_l := (i_{l-1}, i_l)$, $l = 1, \ldots, k$, all belong to O. Since $i_k \notin \{i_0, i_i, \ldots, i_{k-1}\}$, we can find, in view of by Proposition 17, $i_{k+1} \neq i_{k-1}$ such that $p_{k+1} := (i_{k+1}, i_k)$ is in O. Now, one of three things can happen:

(i) $i_{k+1} \in \{i_0, \ldots, i_{k-2}\}$. We set $q_{k+1} = (i_k, i_{k+1})$ and $O' := \{p_1, q_1, p_2, q_2, \ldots, p_{k+1}, q_{k+1}\}$. By construction, the points in O' are distinct and O' is symmetric,. Moreover, q_l and p_{l+1} have the same second coordinate i_l and no other point does. It follows that, for every $(i,j) \in O'$, the two intersections $O' \cap \{(k,l) \colon k = i\}$ and $O' \cap \{(k,l) \colon l = j\}$ both contain precisely two points and, hence, that $S := \{(a_i, a_j) \colon (i,j) \in O'\}$ is a 2-set contained $V_{\partial\tilde{\Omega}}$.

(ii)$i_{k+1} = i_k$. We then set $O' := \{p_0, p_1, q_1, p_2, q_2, \ldots, q_k, p_{k+1}\}$. Again, the points of O' are distinct and O' is symmetric (since $p_{k+1} = (i_k, i_k)$). It follows, as in case (i) above, that $S := \{(a_i, a_j) \colon (i,j) \in O'\}$ is a 2-set contained in $V_{\partial\tilde{\Omega}}$.

(iii) $i_{k+1} \notin \{i_0, i_1, \ldots, i_k\}$. In this case, we set $q_{k+1} = (i_k, i_{k+1})$ and observe that we are back in the situation above with k replaced by $k+1$.

Eventually, this process must stop (since there are only m choices of indices i_l), i.e. for some k one of the two cases (i) or (ii) must occur. In either case, we are left with a 2-set S contained in $V_{\partial\tilde{\Omega}}$. To find a family of 2-sets $S(t)$, $t \in (-\epsilon, \epsilon)$, with the properties described in Proposition 15, we note that we could have picked any $a \in \partial\tilde{\Omega}$, except for at most a finite subset, to start our construction. Moreover, as a varies continuously along $\partial\tilde{\Omega}$, the curves L and L^* can be made to vary continuously. As a consequence, the monodromy action (and hence the group H) will remain the same. It follows that if we let $t \to a(t)$, $t \in (-\epsilon, \epsilon)$, be a parametrization of a piece of the real-analytic curve $\partial\tilde{\Omega}$ such that $a = a(t)$ is a point allowed in the construction, then the resulting family of 2-sets $S(t)$ satisfies the conditions in Proposition 15. $\square$

To complete the proof of Theorem 12, we let $S_X(t)$ be a family of 2-sets on $V_{\partial\tilde{\Omega}} \in X_0 \times X_0^*$ as in Proposition 15. Recall that, except for at most a finite number of points $\zeta_0 \in X$, if $\zeta_1, \ldots, \zeta_n$ are points such that $\Psi(\zeta_j) = \Psi(\zeta_0)$, for $j = 1, \ldots, n$, then the points $\pi(\zeta_j) \in \mathbb{P}$, for $j = 0, \ldots, n$, are all distinct. (Ψ and π generate the function field on X.) Thus, there is a subinterval $I \subset (-\epsilon, \epsilon)$ such that, for $t \in I$, the points $\pi(\zeta_j(t))$, $j = 1, \ldots, n$, are all distinct and different from the point at infinity as are the points $\pi(\tau_j)$, $j = 1, \ldots, n$. Clearly, this means that $S(t) := \Pi(S_X(t))$, where $\Pi(\zeta, \tau)$ is given by (23) and $t \in I$, is a family of 2-sets on $V_{\partial\Omega} \subset \mathbb{C}^2$ with the properties prescribed in Theorem 12. $\square$

5. Proofs of Theorems 5, 6 and 7

In this section, we provide the remaining proofs of the results stated in the introduction.

Proof of Theorem 6. Let us assume, in order to reach a contradiction, that the inverse of the Riemann mapping φ is not rational, but that the solution $u(z, \bar{z})$ to (1), for every v that is the restriction of a polynomial $R(z, \bar{z})$, extends meromorpically to $X \times X^*$. Let X_0, $\tilde{\Omega} \subset X_0$ ($\subset X$) and its complexification $V_{\partial\tilde{\Omega}} \subset X_0 \times X_0^*$ be as in section 4. By Proposition 15, there is a family of 2-sets $S(t) = \{(\zeta_j(t), \tau_j(t))\colon j = 1, \ldots, n\}$, for $t \in (-\epsilon, \epsilon)$, contained in $V_{\partial\tilde{\Omega}}$. Let us renumber the points in the 2-set, if necessary, so that $\tau_j(t) = \tau_{j+1}(t)$ for j odd, and $\zeta_j(t) = \zeta_{j+1}(t)$ for j even. (We pick a starting point and then we traverse the 2-set by alternately moving horizontally and vertically. Once the starting point has been fixed, this can only be done in one way.) Observe that, for each j odd, there is a unique index $k = k(j)$ such that $\zeta_j(t) = \zeta_k(t)$ and, similarly, for j even, there is $k = k(j)$ such that $\tau_j(t) = \tau_k(t)$. We claim that $|j - k(j)|$ is always odd. This is easy to see if we think of the ordering as traversing the 2-set by alternately moving horizontally and vertically. If j is, say, odd (and let us assume, for simplicity, that $j < k(j)$), then our next move is a horizontal one, i.e. $\tau_j(t) = \tau_{j+1}(t)$ and, hence, $\zeta_j(t) \neq \zeta_{j+1}(t)$. After that, we move vertically, i.e. $\zeta_{j+1}(t) = \zeta_{j+2}(t)$ and, hence $\tau_{j+1}(t) \neq \tau_{j+2}(t)$. Clearly, in order to reach a point $(\zeta_k(t), \tau_k(t))$ such that again $\zeta_j(t) = \zeta_k(t)$, our last move must be a horizontal one. Hence, the number of steps $k(j) - j$ we have taken is odd. The other cases are completely analogous. Note, in particular, that j and $k(j)$ are of opposite parity. By using this fact and the fact that $S(t)$ is a 2-set, we conclude that $j \mapsto k(j)$ is a permutation of the set $\{1, \ldots, n\}$.

Let us define a family of discrete measures $\mu(t)$ on $V_{\partial\tilde{\Omega}} \subset X_0 \times X_0^*$ as follows

$$\mu(t) := \sum_{j=1}^{n} (-1)^j \delta_{p_j(t)}, \tag{28}$$

where $p_j(t) := (\zeta_j(t), \tau_j(t))$ and $\delta_{p_j(t)}$ denotes a unit point mass at the point $p_j(t)$. Recall from the end of the proof of Theorem 12 that the points $\Pi(p_j(t)) \subset \mathbb{C}^2$, for $j = 1, \ldots, n$, are all distinct (i.e. $\Pi(S(t))$ is a 2-set) except, possibly, for a finite number of values for t; here $\Pi\colon X_0 \times X_0^* \to \mathbb{C}^2$ is the projection given by (23). We may assume, without loss of generality of course, that $t = 0$ is not one of these excluded values. Let $\nu(t)$ be the discrete measure in $\mathbb{C}^2$ (supported on the 2-set $\Pi(S(t))$ obtained by pushing $\mu(t)$ forward by Π and $R(z, w)$ a polynomial that is not annihilated by $\nu(0)$, i.e.

$$\int R(z,w)d\nu(0) = \sum_{j=1}^{n}(-1)^j R(\Pi(\zeta_j(0)), \overline{\Pi(\tau_j(0))}) \neq 0.$$

(For instance, let $R(z, w)$ be a polynomial that vanishes at all but one point of $\Pi(S(0))$.) It follows that the meromorphic function $\tilde{R}(\zeta,\tau) := R(\Pi(\zeta), \overline{\Pi(\tau)})$ on $X_0 \times X_0^*$ is not annihilated by $\mu(0)$ and, by continuity, not by $\mu(t)$, for t sufficiently small. Let $u(z, \bar{z}) = f(z) + \overline{g(z)}$ be the solution to (1), where v is the restriction to $\partial\Omega$ of $R(z, \bar{z})$. Thus, $f(z) + \overline{g(z)} = R(z, \bar{z})$ on $\partial\Omega$. By assumption, f and g lift to meromorphic functions F and G on X_0 and, for $\zeta \in \partial\tilde{\Omega}$, we have $F(\zeta) + \overline{G(\zeta)} = R(\pi(\zeta), \overline{\Pi(\zeta)})$. By polarization (complexification), we conclude that

$$F(\zeta) + \overline{G(\tau)} = \tilde{R}(\zeta,\tau) \tag{29}$$

for (ζ, τ) on the complexification $V_{\partial\tilde{\Omega}}$. Let us now fix a sufficiently small t_0 such that $\tilde{R}(\zeta,\tau)$ is not annihilated by $\mu(t_0)$ and such that F has no poles at the points $\zeta_j(t_0)$, for $j = 1, \ldots n$, and G has no poles at the points $\tau_j(t_0)$, for $j = 1, \ldots n$. (This can be done, since $t \mapsto \zeta_j(t)$ and $t \mapsto \tau_j(t)$ trace out non-trivial piecewise real-analytic curves.) We claim that $F(\zeta) + \overline{G(\tau)}$ is annihilated by $\mu(t_0)$. Indeed, if we use the notation $\mu = \mu(t_0)$, $\zeta_j = \zeta_j(t_0)$, and $\tau_j = \tau_j(t_0)$, then we have

$$\begin{aligned}
\int \big(F(\zeta) + \overline{G(\tau)}\big)d\mu =& \sum_{j=1}^{n}(-1)^j\big(F(\zeta_j) + \overline{G(\tau_j)}\big)\\
=& \frac{1}{2}\sum_{j=1}^{n}(-1)^j\big(F(\zeta_j) + \overline{G(\tau_j)}\big) +\\
& \frac{1}{2}\sum_{j=1}^{n}(-1)^{k(j)}\big(F(\zeta_{k(j)}) + \overline{G(\tau_{k(j)})}\big)\\
=& 0.
\end{aligned} \tag{30}$$

The second identity follows from the fact that $j \mapsto k(j)$ is a permutation and the third from the fact that j and $k(j)$ have opposite parity. This is a contradiction, in view of (29), and hence the proof is complete. □

Proof of Theorem 7. We first observe that if $u(z, \bar{z})$ extends as a period free harmonic function in $\mathbb{C} \setminus A$, then there are holomorphic functions f and g in $\mathbb{C} \setminus A$ such that $u(z, \bar{z}) = f(z) + \overline{g(z)}$. To see this, observe that the function $v(z, \bar{z}) := \int_{z_0}^{z} *du$, where z_0 is any fixed point and the integration takes place along any curve in $\mathbb{C}\setminus A$ connecting z_0 to z, is well defined. Moreover, it is easy to see that $v(z,\bar z)$ is a conjugate harmonic function to u (i.e. u and v are related by the Cauchy-Riemann equations). It follows that $f = u + iv$ and $g = \bar u + i\bar v$ are holomorphic functions in $\mathbb{C}\setminus A$ and that $u(z,\bar z) = f(z) + \overline{g(z)}$.

We first claim, under the hypotheses of the theorem, that the inverse of the Riemann mapping φ must be rational. Assume, in order to reach a contradiction, that φ^{-1} is not rational. Let $S(t)$ be a family of 2-sets in the complexified boundary $V_{\partial\Omega}$ as given by Theorem 12. Construct a family of discrete measures $\mu(t)$ in $\mathbb{C}^2$ as in the proof of Theorem 6 (cf. (28)), let $R(z,\bar z)$ be any polynomial that is not annihilated by the measure $\mu(0)$, and let $u(z,\bar z) = f(z) + \overline{g(z)}$ be the solution to the Dirichlet problem (1), where v is the restriction of $R(z,\bar z)$. Since $f(z) + \overline{g(z)} = R(z,\bar z)$ on $\partial\Omega$ and f and g extend, by assumption, holomorphically to $\mathbb{C}\setminus A$, for some discrete set A, we may proceed as in the proof of Theorem 6 above to reach the desired contradiction. We leave the details of this to the reader.

To complete the proof, we must show, under the hypotheses in the theorem and the additional hypothesis that φ^{-1} is rational, that Ω actually must be a disk. Let $S(z)$ be the Schwarz function of $\partial\Omega$ as in section 3. Since φ^{-1} is rational, $S(z)$ is an algebraic function that extends as a meromorphic function to a neighborhood of $\overline{\Omega}$ (see e.g. [S92]). Let $P(z)$ be a polynomial such that $f(z) = P(z)S(z)$ is holomorphic in Ω. Note that, on the boundary, $f(z) = P(z)\bar z$. It follows that $u(z,\bar z) = f(z)$ is the solution to the Dirichlet problem (1), where v is the restriction of the polynomial $R(z,\bar z) := P(z)\bar z$. Since $f(z)$, by assumption, extends as a holomorphic function in $\mathbb{C}\setminus A$, it follows that $S(z)$ extends as a meromorphic function to $\mathbb{C}\setminus A$. However, since $S(z)$ is also algebraic and A is discrete, $S(z)$ must in fact be rational. This proves that Ω is a disk by Davis' theorem (as in section 3; see [D74]). This completes the proof. □

Proof of Theorem 5. The implication (ii) $\Longrightarrow$ (i) follows from Theorem 6. To prove the opposite implication, we recall a result from [E92] stating that the solutions to the Dirichlet problems considered in Theorem 5 lift to meromorphic functions on $Y\times Y^*$, where Y denotes the Riemann surface of the Schwarz function $S(z)$ of $\partial\Omega$ and Y^* the conjugate Riemann surface. Recall that

$$S(z) = \varphi^{-1}(1/\overline{\varphi(z)}).$$

Since φ^{-1} is rational, it follows that the Riemann surface Y coincides with the Riemann surface X of φ. Consequently, the implication (i) $\Longrightarrow$ (ii) is a direct consequence of the above mentioned result from [E92]. □

6. Concluding remarks

In this final section, we shall address a few questions that arose during the completion of this paper. We shall also briefly indicate the relation between our construction of 2-sets (in Section 4 above) and the notion of lightning bolts in approximation theory. The material below is arranged in three separate subsections.

6.1. **2-sets and lightning bolts.** The notion of a 2-set (introduced in Section 4) is closely related to the notion of a closed lightning bolt in $\mathbb{R}^n$ introduced by Arnold and Kolmogorov to study Hilbert's thirteenth problem on expressing a function in n variables as a superposition of functions of fewer variables. We shall refer to [Kh97] for the history of the problem, detailed discussions and relevant references. Here we just very briefly sketch how this notion applies to our situation and the proof of Theorem 6.

A "complex" lightning bolt is a finite set of points (vertices) $p_0, q_0, p_1, \dots, p_n, q_n$ in $\mathbb{C}^2$ such that each complex line connecting p_j to q_j or q_j to p_{j+1} is either "horizontal" or "vertical", i.e. has either its first or second coordinate fixed. A lightning bolt is said to be irreducible if it does not contain a lightning bolt with smaller number of vertices still connecting the first and last vertex (p_0 and q_n). A lightning bolt is closed if $p_0 = q_n$. Every closed lightning bolt, as is easily seen, has an even number of vertices and supports a finite measure μ (defined by (28)) consisting of charges with alternating signs at the vertices. Also, it is obvious that a 2-set is an irreducible closed lightning bolt while every closed lightning bolt is a finite union of 2-sets. The measure (28) is (cf. the proof of Theorem 6 above) an annihilating measure for all holomorphic functions in $\mathbb{C}^2$ representable in the form $f(z) + g(w)$. Therefore if a variety V supports a closed lightning bolt, there exists a vast set of functions, holomorphic in a neighborhood of V (even polynomials!), that cannot be approximated by sums of (holomorphic) functions of one variable, $f(z)+g(w)$. Our construction in Section 4 precisely produces on the variety V, a connected component of the complexified boundary of the domain Ω, a closed irreducible lightning bolt that carries a measure annihilating all functions $f(z) + g(w)$, with f, g, holomorphic in a neighborhood of V. In fact, already the existence of a closed lightning bolt on V would be sufficient, but of course, it is not hard to see that every closed lightning bolt contains an irreducible lightning bolt, hence contains a 2-set that we constructed directly in Section 4. Essentially, the technical subtlety of the construction reduces to the following; since V, as in Section 4, represents a Riemann surface of degree at least 2, we could, starting at any non-critical point p of V construct a lightning bolt by simply going on a horizontal $\{z = z_0\}$, or vertical $\{w = w_0\}$ line from p until we hit V again and then proceed at each step changing the "type" of the line emanating from a newly obtained vertex to the opposite from the type of the complex line on which we have arrived at the vertex, of course, avoiding critical values and critical points of V, a finite set. The difficulty is to show that the process will terminate rather than produce a lightning bolt with infinitely many vertices running away to infinity. This is why we needed the specific construction of a rather special family of grids of points ($S(t)$ on V in Section 4) obtained as orbits of a special finite subgroup of the monodromy group with two generators, to prevent an associated lightning bolt "running away" to infinity.

In view of the above discussion, we could in fact infer the following corollary from our construction in Section 4.

Corollary 19. *Let Ω be a smoothly bounded simply connected domain in the plane satisfying the hypotheses in Theorem 12. Then, there exist functions holomorphic in a neighborhood U of V in $\mathbb{C}^2$ that cannot be approximated uniformly on compact subsets of V by functions of the form $f(z)+g(w)$, with f, g holomorphic in U.*

Finally, note that the real variable version of our problem is also of interest. Indeed, assume that we are interested in investigating whether every continuous function on a closed Jordan curve Γ can be uniformly approximated by functions of the form $f(x)+g(y)$, $f, g \in C(\Gamma)$, i.e. by solutions of the two-dimensional wave equation $\partial^2 u/\partial x \partial y = 0$. Then, in view of Schnerel'man's theorem, Γ always supports vertices of a rectangle. Hence, by rotating Γ, we obtain on its image Γ' four vertices of a rectangle whose sides are parallel to coordinate axes, i.e. Γ' carries a closed irreducible lightning bolt and the answer to the above question is a resounding "no" , but for Γ' rather than for original curve Γ. Of course, for convex curves symmetric with respect to coordinate axes, it is rather obvious that there exists a rectangle with vertices on the curve whose sides are parallel to the axes, e.g., consider an ellipse. For most curves, however, the answer to the above question is still not known; cf. [Kh97] for more detailed discussion and further references.

6.2. **Irreducibility of the variety $\{\varphi(z)\overline{\varphi(\bar{w})}=1\}$.** In this section, we shall address a question that arises naturally in our proof of Theorem 12. For clarity, we shall confine our discussion to the situation where a Riemann map $\varphi\colon \Omega \to \mathbb{D}$ is rational. Recall that the complexification of $\partial\Omega$ is denoted by $V_{\partial\Omega} \subset \mathbb{C}^2$. As observed in the proof of Theorem 12, $V_{\partial\Omega}$ is an irreducible component of the algebraic variety

$$V := \big\{(z,w)\colon \varphi(z)\overline{\varphi(\bar{w})}-1=0\big\}. \tag{31}$$

We remark that V can also be expressed as follows

$$V := \big\{(z,w)\colon p(z)\overline{p(\bar{w})}-q(z)\overline{q(\bar{w})}=0\big\}, \tag{32}$$

where $\varphi(z)=p(z)/q(z)$ and p and q are relatively prime. One may ask: *Under what conditions is the algebraic variety in* (31) *irreducible*? It is easy to see that V need not be irreducible if we do not assume that φ is a biholomorphism $\Omega \to \mathbb{D}$. For instance, suppose that $r\colon \Omega \to \mathbb{D}$ is a rational map and consider $\varphi(z)=(r(z))^2$. Clearly,

$$\varphi(z)\overline{\varphi(\bar{w})}-1 = r(z)^2\overline{r(\bar{w})}^2-1 = (r(z)\overline{r(\bar{w})}-1)(r(z)\overline{r(\bar{w})}+1).$$

The following, less trivial example shows that irreducibility of V may fail even when $\varphi\colon \Omega \to \mathbb{D}$ is biholomorphic if we allow corners in the boundary of Ω. (Observe, however, that in this example the boundary of Ω is not contained in an irreducible real-algebraic variety and, hence, the complexification is not irreducible.)

Example 1. Let Ω be the cigar-shaped domain given by

$$\Omega := \big\{z\colon |z - i/\sqrt{2}| < 1,\ |z + i/\sqrt{2}| < 1\big\}. \tag{33}$$

The domain Ω is bounded by two circular arcs meeting at right angles at the two points $z = \pm 1/\sqrt{2}$. We leave it to the reader to verify that the mapping $w = \varphi(z)$, where $\varphi := \varphi_3 \circ \varphi_2 \circ \varphi_1$ and

$$\varphi_1(z) := \frac{e^{i\pi/4}}{z - 1/\sqrt{2}}, \quad \varphi_2(z) := z^2, \quad \varphi_3(z) := \frac{z-i}{z+i}, \tag{34}$$

is a biholomorphic mapping $\Omega \to \mathbb{D}$. Clearly, φ is rational. Moreover, the variety V given by (31) cannot be irreducible, since it contains the complexifications of both circles $|z - 1/\sqrt{2}| = 1$ and $|z + 1/\sqrt{2}| = 1$.

In the example above, the degree of the rational Riemann mapping is two, and the number of distinct irreducible components of the variety V is at least two. As we shall see below, it is the corners on the boundary that make it possible for V, in the example, to have more than one component. Indeed, for any *smoothly* bounded domain with a degree two rational Riemann mapping, the variety (31) is irreducible (see Corollary 21 below). To see this, we shall first establish a bound on the possible number of components of V under an additional condition on φ. (The condition implies that, after applying suitable Möbius transformations, we may assume that φ is a polynomial mapping; see the proof below.)

Proposition 20. *Let $\varphi(z)$ be rational and assume there is a point $b \in \mathbb{P} \setminus \partial\mathbb{D}$ such that $\varphi^{-1}(b) \subset \mathbb{P}$ contains only one distinct point. Let $b^* := 1/\bar{b} \in \mathbb{D}$ and denote by k the number of distinct points in $\varphi^{-1}(b^*)$. Then, the number n of components of V, where V is given by* (31)*, counted with multiplicities satisfies the following inequality*

$$n \leq \frac{\deg \varphi}{k}. \tag{35}$$

Proof. We regard φ as a holomorphic mapping $\mathbb{P} \to \mathbb{P}$. Let $\zeta \in \mathbb{P}$ be the distinct point in $\varphi^{-1}(b)$. Let ψ_1 be a Möbius transformation of the source copy of $\mathbb{P}$ sending ζ to ∞, and ψ_2 a Möbius transformation of the target copy of $\mathbb{P}$ sending b to ∞ and preserving $\partial\mathbb{D}$. Observe that ψ_2 sends $b^* = 1/\bar{b}$ to 0. Now, the mapping $p := \psi_2 \circ \varphi \circ \psi_1 \colon \mathbb{P} \to \mathbb{P}$ is a polynomial (since $p^{-1}(\infty) = \{\infty\}$) with $\deg p = \deg \varphi$, and the number k of distinct points in $\varphi^{-1}(b^*)$ equals the number of distinct point in $p^{-1}(0)$. The variety $W := \{(z, w)\colon p(z)\overline{p(\bar{w})} - 1 = 0\}$ is birationally equivalent to V and hence W and V have, in particular, the same number of irreducible components, namely n. Thus, to prove the proposition, it suffices to prove the estimate

$$n \leq \frac{\deg p}{k}, \tag{36}$$

where n is the number of irreducible components of W counting multiplicites and k is the number of distinct zeros of p. We let $A_j(z, w)$, for $j = 1, \ldots, n$, be irreducible polynomials defining the n components of W, repeated according to the multiplicities of the components. If we let $a_1, \ldots, a_k$ be the distinct zeros of p and let $r_1, \ldots, r_k$ denote their respective multiplicities, then we have, for some constant c,

$$c \prod_{j=1}^{k} (z - a_j)^{r_j} (w - \bar{a}_j)^{r_j} - 1 = \prod_{l=1}^{n} A_l(z, w). \tag{37}$$

Observe that no $A_l(z, w)$ can be a polynomial of z (or w) alone. (If, say, A_1 were a polynomial of z, then the left hand side would vanish identically for every $z = z_0$ such that $A_1(z_0) = 0$. Clearly, this cannot happen.) If we substitute $z = a_1$ in (37), then we obtain

$$-1 = \prod_{l=1}^{n} A_l(a_1, w).$$

Since the $A_l(a_1, w)$ are polynomials, we conclude that each $A_l(a_1, w)$ must be constant, i.e.

$$A_l(z, w) = c_l^0 + (z - a_1) A_l^1(z, w), \tag{38}$$

for some constant c_l^0 and polynomial $A_l^1(z, w)$. Note that $A_l^1(z, w)$ cannot be constant since $A_l(z, w)$ is not a polynomial of z alone. Next, we substitute $z = a_2$ in (37) and conclude, in the same way, that each $A_l(a_2, w)$ must be constant. Since $a_1 \neq a_2$, we conclude from (38) that $A_l^1(a_2, w)$ is constant, i.e.

$$A_l^1(z, w) = c_l^1 + (z - a_2) A_l^2(z, w),$$

for some constant c_l^1 and polynomial $A_l^2(z, w)$. It follows that

$$A_l(z, w) = c_l^0 + c_l^1 (z - a_1) + (z - a_1)(z - a_2) A_l^2(z, w). \tag{39}$$

Again, $A_l^2(z, w)$ cannot be constant, since $A_l(z, w)$ is not a polynomial of z. We may proceed in the obvious way, substituting $z = a_j$ for $j = 3, \ldots, k$ in (37). In the end, we will obtain, for each $l = 1, \ldots, n$,

$$A_l(z, w) = c_l^0 + \sum_{j=1}^{k-1} c_l^j \prod_{i=1}^{j} (z - a_i) + A_l^k(z, w) \prod_{i=1}^{k} (z - a_i), \tag{40}$$

where $c_l^0, \ldots, c_l^{k-1}$ are constants and $A_l^k(z, w)$ is a nonconstant polynomial. In particular, $A_j(z, w)$ has degree at least k in z. It follows that the product on the right in (37) has degree at least nk in z. On the other hand, the left hand side has degree $\deg p$ in z. Thus, $nk \leq \deg p$. This proves the estimate (36) and, hence, the proposition. $\square$

As a corollary, we obtain the following result.

Corollary 21. *Let Ω be a simply connected, bounded domain with smooth boundary. Suppose that a Riemann mapping $\varphi\colon \Omega \to \mathbb{D}$ is rational with $\deg \varphi \leq 2$. Then, the variety V, given by (31), is irreducible.*

Proof. We may assume that $\deg \varphi = 2$, since the case $\deg \varphi = 1$ is trivial (φ is a Möbius transformation). Since Ω has smooth boundary, the rational function φ extends as a biholomorphism of a neighborhood of $\overline{\Omega}$ to a neighbborhood of $\overline{\mathbb{D}}$. It follows that $\varphi^{-1}(\mathbb{D})$ consists of two disjoint components Ω and Ω', and φ is a biholomorphism also of a neighborhood of $\overline{\Omega'}$ to a neighborhood of $\overline{\mathbb{D}}$. We conclude that $G := \varphi^{-1}(\mathbb{P} \setminus \overline{\mathbb{D}})$ is connected but not simply connected, and $\varphi\colon G \to \mathbb{P} \setminus \overline{\mathbb{D}}$ is 2-to-1. Hence, $\varphi\colon G \to \mathbb{P} \setminus \overline{\mathbb{D}}$ must be branched, i.e. there is a $\zeta \in G$ such that ζ is a root of multiplicity at least two of the equation $\varphi(z) = b$ with $b := \varphi(\zeta) \in \mathbb{P}\setminus\overline{\mathbb{D}}$. Since φ has degree two, we conclude that $\varphi^{-1}(b)$ consists only of the point ζ. Thus, we may apply Proposition 20. The point $b^* = 1/\bar{b}$ is in $\mathbb{D}$ and, consequently, the set $\varphi^{-1}(b^*)$ consists of two points (one in Ω and one in Ω'), i.e. $k = 2$. The estimate (35) implies that the number n of components is ≤ 1 and, hence, V is irreducible. $\square$

As a side remark, we mention that Corollary 21, in view of the discussion in the opening paragraph of Section 4, can also be directly deduced from Corollary 14; we leave the details of this to the interested reader. The assumption that $\partial\Omega$ is smooth in Corollary 21 is necessary in view of Example 1. We should also point out that the rational mapping φ in Example 1 does not satisfy the hypotheses of Proposition 20, since the only points b for which $\varphi^{-1}(b)$ consists of a single point lie on the boundary $\partial\mathbb{D}$.

We conclude this subsection by pointing out that we do not know of any examples where V, given by (31), is reducible (i.e. has more than one irreducible component) when $\partial\Omega$ is *smooth* (necessarily real-analytic in this case; in particular, its complexification is irreducible) and $\varphi\colon \Omega \to \mathbb{D}$ is a rational biholomorphic mapping.

6.3. **Cusps and poles.** In Lemma 10, it was shown that if Ω is simply connected, a harmonic function $u(z, \bar{z}) = f(z) + \overline{g(z)}$ extends continuously to $\partial\Omega$, and f, g extend meromorphically across $\partial\Omega$, then f and g cannot have poles on $\partial\Omega$ except possibly at outward pointing cusps on $\partial\Omega$. This result suffices for the proof of Theorem 2 since outward pointing cusps can be ruled out in that situation. As mentioned in the remark following Lemma 10, however, poles can actually occur at cusps in the general case. For completeness, we give here a result that shows that poles cannot occur at cusps under a stronger condition on the boundary data of the harmonic function.

Proposition 22. *Let Ω be a bounded simply connected domain in $\mathbb{C}$. Let U be an open disk that intersects the boundary of Ω and assume that $\Gamma := \partial\Omega \cap U$ is a piecewise smooth curve. Suppose that $u(z, \bar{z})$ is a harmonic function that extends continuously up to Γ and coincides there with a function $R(z, \bar{z})$ that is smooth in U. If f and g are holomorphic* *functions in Ω such that $u(z,\bar{z}) = f(z) + \overline{g(z)}$, and f and g extend meromorphically to U, then f and g cannot have poles on Γ.*

Proof. In view of Lemma 10, it suffices to show that f and g do not have poles at outward pointing cusps on Γ. Let z_0 be such a cusp and assume, in order to reach a contradiction, that f and g extend meromorphically to a full neighborhood of z_0 with a pole of order $p \geq 1$ at z_0. (As in the proof of Lemma 10, it is easy to see that if one of the functions has a pole at z_0, then the other function must, by the continuity of u up to Γ, have a pole of the same order at that point.) After a translation and rotation if necessary, we may assume that $z_0 = 0$, that $\Omega \cap U$ is contained in $\{\mathrm{Re}\ z > 0\}$, and that the common tangent of the two curves making up Γ near z_0 is the x-axis. The piecewise smooth curve Γ near $z = 0$ is made up of two curves $\gamma_j : [0, \epsilon) \to \mathbb{C}$, for $j = 1, 2$, such that $\gamma_j(0) = 0$ and $\gamma_j'(0) = 1$. Let $h(x)$ be a function, defined for $x \geq 0$, such that one of these curves, say $\gamma := \gamma_1$, is given by the graph $y = h(x)$, for $x \geq 0$. We note that the function $h(x)$ is smooth for $x > 0$ and, since Γ has a cusp pointing along the negative x-axis, $h(x)$ is C^1 up to $x = 0$ with $h(0) = h'(0) = 0$. (We remark that in general, even if Γ is piecewise real-analytic, the function will not be better than C^1 up to $x = 0$; consider e.g. the cusp $y^2 - x^3 = 0$.)

Let us expand $f(z)$ and $g(z)$ in their Laurent series at $z = 0$ as in (15). The arguments in the proof of Lemma 10 show that the leading coefficients a and b in (15) must be related by $a = -b$ (cf. (19) with $\zeta = 1$). Hence, we can write $u(z,\bar{z})$ as follows

$$\begin{aligned} u(z,\bar{z}) &= z^{-p} - \bar{z}^{-p} + \sum_{j=1}^{p-1} (a_j z^{-j} + b_j \bar{z}^{-j}) + v(z,\bar{z}) \\ &= \frac{1}{|z|^{2p}} \left(\bar{z}^p - z^p + \sum_{j=1}^{p-1} (a_j z^{p-j} \bar{z}^p + b_j z^p \bar{z}^{p-j}) \right) + v(z,\bar{z}) \end{aligned} \tag{41}$$

where $v(z,\bar{z})$ is real-analytic in a neighborhood $z = 0$. The fact that $u(z,\bar{z})$ and $R(z,\bar{z})$ coincide on γ means, of course, that $u(x+ih(x), x-ih(x)) = R(x+ih(x), x-ih(x))$ for $x \geq 0$. We may rewrite this relation as follows

$$\begin{aligned} &(x - ih(x))^p - (x + ih(x))^p + p(x, h(x)) \\ &\qquad = (x^2 + h(x)^2)^p \big(R(x+ih(x), x-ih(x)) - v(x+ih(x), x-ih(x)) \big), \end{aligned} \tag{42}$$

where $p(x,y)$ is the polynomial given by

$$p(x,y) := \sum_{j=1}^{p-1} \big(a_j (x+iy)^{p-j} (x-iy)^p + b_j (x+iy)^p (x-iy)^{p-j} \big) \tag{43}$$

Observe that $p(x, xt) = x^{p+1} q(x,t)$, where $q(x,t)$ is a polynomial. Recall that $h(x)$ is C^1 up to $x = 0$ with $h(0) = h'(0) = 0$. Thus, we may write $h(x) = xk(x)$, where $k(x)$ is continuous up to $x=0$ with $k(0)=0$. If we substitute $h(x)=xk(x)$ in (42) and cancel a factor x^p, we obtain

$$(1-ik(x))^p-(1+ik(x))^p+xq(x,k(x)) - x^p(1+k(x)^2)^p\big(R(x+ixk(x),x-ixk(x))-v(x+ixk(x),x-ixk(x))\big)=0. \tag{44}$$

This means that $t=k(x)$ solves the equation

$$F(x,t)=0, \tag{45}$$

where

$$F(x,t):=(1-it)^p-(1+it)^p+xq(x,t) - x^p(1+t^2)^p\big(R(x+ixt,x-ixt)-v(x+ixt,x-ixt)\big). \tag{46}$$

Observe that $F(x,t)$ is smooth in a neighborhood of $(x,t)=(0,0)$, $F(0,0)=0$, and

$$\frac{\partial F}{\partial t}(0,0)=-2ip.$$

By the implicit function theorem, the equation $\operatorname{Im} F(x,t)=0$ has a unique smooth solution $t=k(x)$ with $k(0)=0$. Recall also that the solution obtained by the implicit function theorem is actually unique among all possible, say continuous, solutions. Since the function $F(x,t)$ only depends on the data $R(z,\bar z)$ and the solution $u(z,\bar z)=f(z)+\overline{g(z)}$, we conclude that if f and g extend meromorphically with a pole of order $p\geq 1$ at 0, then u and R can only coincide on one *single* curve $y=h(x)$, where $h(x)=xk(x)$ and $t=k(x)$ is the unique solution of $F(x,t)=0$, through the origin. This contradicts the fact that u and R coincide on two curves that form a cusp at $z=0$ and, hence, completes the proof. □

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