Position Dependent NLS Hierarchies: Involutivity, Commutation Relations, Renormalisation and Classical Invariants

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Abstract

We consider a family of explicitly position dependent hierarchies $(I_n)_0^\infty$, containing the NLS (non-linear Schrödinger) hierarchy. All $(I_n)_0^\infty$ are involutive and fulfill $D I_n = n I_{n-1}$, where $D = D^{-1} V_0$, $V_0$ being the Hamiltonian vector field $v^\delta_\partial q - u^\delta_\partial q$ afforded by the common ground state $I_0 = uv$. The construction requires renormalisation of certain function parameters.

It is shown that the ‘quantum space’ $\mathbb{C}[I_0, I_1, \ldots]$ projects down to its classical counterpart $\mathbb{C}[p]$, with $p = I_1/I_0$, the momentum density. The quotient is the kernel of $D$. It is identified with classical semi-invariants for forms in two variables.

Introduction: Consider in 1+1 dimensions the (free) heat equation system ($u$ and $v$ are functions of time, $t$, and space $q$)

$$\dot{u} + \frac{1}{2} u'' = 0; \quad -\dot{v} + \frac{1}{2} v'' = 0.\quad (1)$$

With appropriate ‘boundary conditions’ on $u$ and $v$ (e.g. rapid decrease at infinity or periodicity), all $I_n := \frac{1}{2} (u^{(n)} v + (-1)^n u v^{(n)})$ are conservation laws:

$$\frac{d}{dt} \int I_n \, dq = 0, \quad n = 0, 1, 2, \ldots \quad (2)$$
This is an immediate consequence of the equations being invariant under space translations. There is an additional first order conservation law, viz. \( tI_1 - qI_0 \).

The counterpart of \( (I_n)^\infty_0 \) for the free classical (Newton) equation \( \ddot{q} = 0 \) is the sequence \( (p^n)^\infty_0 \), of constants of motion. (\( p = \dot{q} \), as usual.) Obviously, all \( p^n \) commute in the Poisson bracket

\[
\{\xi, \eta\} := \frac{\partial \xi}{\partial p} \frac{\partial \eta}{\partial q} - \frac{\partial \xi}{\partial q} \frac{\partial \eta}{\partial p}
\]

(3)

The additional first order (in \( p \)) constant of motion \( pt - q \) satisfies

\[
\{pt - q, p^n\} = np^{n-1} = dp^n/dp.
\]

(4)

(\( t \) is looked upon as a parameter.)

Similarly, with the (field theory) bracket \( \{F, G\} := \int \left( \frac{\delta F}{\delta u} \frac{\delta G}{\delta v} - \frac{\delta F}{\delta v} \frac{\delta G}{\delta u} \right) dq \), we have

\[
\{tI_1 - qI_0, I_n\} = \{-qI_0, I_n\} =: DI_n = nI_{n-1}.
\]

(5)

This is of course related to \( \langle 1, p, pt - q \rangle \) and \( \langle I_0, I_1, tI_1 - qI_0 \rangle \), respectively, being representations of the Heisenberg algebra.

Suppose now that we form \( \mathbb{C}[I_0, I_1, I_2, \ldots] \): all polynomials in the variables \( I_0, I_1, I_2, \ldots \). What, if any, is the relation to the classical version, viz. \( \mathbb{C}[p] \), all polynomials in \( p \)?

Below it is shown that there is a projection

\[
\mathbb{C}[I_0, I_1, I_2, \ldots] \to \mathbb{C}[I_1/I_0] \simeq \mathbb{C}[p]
\]

(6)

with ‘fibre’ ker \( D \), which in its turn is related to the classical 19th century semi-invariants of Cayley and others. See Gurevich [9], Ibragimov [10], Olver [27, 28].

The paper is devoted to this and some related questions, among them renormalisation, for a wider class of commuting conservation laws, containing a version of the non-linear Schrödinger hierarchy, NLS.

As background serve the papers on invariance properties, including behaviour under mappings between manifolds, for Schrödinger and related diffusion processes [4, 6, 14, 16, 17, 18, 22, 30, 31], in particular the case of Gaussian diffusions [2, 3, 18]. At the centre of much of this is the heat Lie algebra, first described by Lie in 1881 [21]. See e.g. Ibragimov [10, 11, 12] and Olver [28]. Other general background references are [2], [6] and [19, 20], and for the NLS equation primarily [7], together with [23, 24] and [32].
1 Outline and formulation of results

Consider all \( C^\infty \) curves \((u(q), v(q)), q \in \mathbb{R}\), in \( \mathbb{C}^2 \). We are interested in functionals or differential functions of the form \( F = f(q, u, u_1, \ldots, u_n; v, v_1, \ldots, v_n) \), where \( u_j = u^{(j)}, v_j = v^{(j)} \) and where it is understood that all the \( u_i \) and \( v_j \) depend on \( q \), the coordinate in the base space. Here \( f \) is \( C^\infty \) in the appropriate space, a jet bundle. With \( D = d/dq \) we form variational derivatives:

\[
\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - D \frac{\partial F}{\partial u_1} + D^2 \frac{\partial F}{\partial u_2} + \ldots, \quad \frac{\delta F}{\delta v} = \frac{\partial F}{\partial v} - D \frac{\partial F}{\partial v_1} + D^2 \frac{\partial F}{\partial v_2} + \ldots \tag{7}
\]

The variational gradient \( \delta F \) of \( F \) is the transpose of the vector \( (\delta F/\delta u, \delta F/\delta v) \). Two functionals \( F \) and \( G \) are identified whenever \( \delta(F - G) = 0 \). This is equivalent to saying that \( F - G \in \text{im} D \). The interpretation is that we have put extra ‘gauge’ conditions on \( u \) and \( v \), e.g. on their behaviour at infinity.

The bracket is, when emphasising the Hamiltonian densities \( F \) and \( G \),

\[
\{F, G\} := \frac{\delta F}{\delta u} \frac{\delta G}{\delta v} - \frac{\delta F}{\delta v} \frac{\delta G}{\delta u} \pmod{\text{im} D}. \tag{8}
\]

We will also use the more customary representation \( \int \left( \frac{\delta F}{\delta u} \frac{\delta G}{\delta v} - \frac{\delta F}{\delta v} \frac{\delta G}{\delta u} \right) dq \) of the bracket. In this picture the central objects are Hamiltonians \( \int F, \int G \ldots \)

**Remark:** Everything we do here could be done for for general elements \( u \) and \( v \) in a commutative algebra, not necessarily \( C^\infty(\mathbb{R}) \), with a derivation.

We shall consider sequences of functionals \( I_0, I_1, I_2, \ldots \) given by a (recursion or) creation operator \( C \):

\[
(i) \quad I_n = CI_{n-1} = C^n I_0, \quad n \geq 0,
\]

or, infinitesimally, \( \delta I_n = C\delta I_{n-1} \). Throughout this paper, we will have

\[
I_0 = uv. \tag{9}
\]

The operator \( D \) given by

\[
D F := D^{-1} \left( v \frac{\delta F}{\delta v} - u \frac{\delta F}{\delta u} \right) = \{-qI_0, F\}, \tag{10}
\]

is well defined on the space \( \overline{A} \) of equivalence classes of functionals that commute with \( I_0 \). We want the \( I_n \) to satisfy

\[
(ii) \quad DI_n = nI_{n-1}, \quad n \geq 0.
\]
Together, properties (i) and (ii) yield a representation of the Heisenberg algebra: we have $[D, C] = 1$ (the identity) on $\bigoplus_{n \geq 0} CI_n$. $D$ is the annihilation operator. There are traces of (ii) in Dickey’s book [5], in connection with the KdV equation.

We also want the $I_n$ to be involutive, i.e. to commute:

$$(iii) \quad \{I_n, I_m\} = 0, \quad \text{all } n, m \geq 0.$$  

Properties (ii) and (iii) imply that the expected value of position, $q$, taken in the (ground) state $I_0$,

$$\langle q \rangle = \int qI_0 \, dq, \quad (11)$$

fulfills the free Newton equations

$$\frac{d^2\langle q \rangle}{dt_n dt_m} = 0, \quad \text{all } n, m \geq 0. \quad (12)$$

Here $t_n$ is the ‘time’ obtained using the Hamiltonian $I_n$. Property (iii) means that for any $m, n$, $dI_n/dt_m = 0$ in the space of equivalent functionals: each $I_n$ is a conservation law w.r.t. any choice of time $t_m$.

Define an auxiliary creation operator $\hat{C}$ by

$$\hat{C}\delta F := \left( \frac{\delta \hat{C} F}{\delta u}, \frac{\delta \hat{C} F}{\delta v} \right)^T = \begin{pmatrix} -D & 0 \\ 0 & D \end{pmatrix}\delta F - 2\lambda \begin{pmatrix} v \\ u \end{pmatrix}DF, \quad (13)$$

where $\lambda$ is a (real or complex) parameter. (This is a slight adaption of [8].)

Let the sequence of functionals $\hat{I}_n$ be given by

$$\hat{I}_n = \hat{C}^n I_0, \quad n \geq 0. \quad (14)$$

Consider two special cases:

$\lambda = 0$ leads to the free case (we write $D^\dagger = -D$)

$$\hat{I}_n(0) = \frac{1}{2}(u_n v + (-1)^n uv_n) = \frac{1}{2}(D^nu \cdot v + uD^ln v). \quad (15)$$

For real non-zero $\lambda$, say $\lambda = 1$, we get a version of the NLS (non-linear Schrödinger) hierarchy (Faddeev-Takhtajan [8]). $\hat{I}_0 = I_0$ and $\hat{I}_1$ are the
same, whereas the next few are

\[ \hat{I}_2 = \hat{I}_2(0) - u^2v^2, \quad \hat{I}_3 = \hat{I}_3(0) - \frac{3}{2}uv(u_1v - uv_1), \]
\[ \hat{I}_4 = \hat{I}_4(0) - uv(u_2v + uv_2) + 4uu_1v_1v + 2u^3v^3, \]
\[ \hat{I}_5 = \hat{I}_5(0) + 5uv(u_2v_1 - u_1v_2) + 5u^2v^2(u_1v - uv_1), \]
\[ \hat{I}_6 = \hat{I}_6(0) - 3(uu_2v_1^2 + u_2^2v_1v_2) - 12uu_2vv_2 + 5u^2v^2 \]
\[ - (u_2^2v^2 + u^2v_2^2) - 50u^2u_1v^2v_1 - 10uv(u_1^2v^2 + u^2v_1^2) - 5u^4v^4. \]

Here, \( \hat{I}_2 \) is the Hamiltonian for the NLS equations. \( I_3 \) leads to KdV upon putting \( v \equiv 1 \). The entire KdV hierarchy can be deduced from the odd-indexed \( \hat{I}_n \).

We introduce an extended family \( I_n = I_n(\lambda, \phi), n \geq 0 \) as follows: Let \( \phi \in C^\infty(\mathbb{R}) \) and put

\[ \Lambda = D + \phi, \quad \Lambda^\dagger = -D + \phi. \] (16)

We note in passing that \( [\Lambda, \Lambda^\dagger] = 2\phi' \) (as a multiplication operator). The case when \( \phi = q \) (or a first order polynomial in \( q \)) gives the Heisenberg algebra.

We define

\[ C\delta F = \begin{pmatrix} \Lambda^\dagger & 0 \\ 0 & \Lambda \end{pmatrix} \delta F - 2\lambda \begin{pmatrix} v \\ u \end{pmatrix} DF \] (17)

with the above requirement on \( F \). One finds

\[ I_1 = \hat{I}_1 + \phi \hat{I}_0 \quad \text{and} \quad I_2 = \hat{I}_2 + 2\phi \hat{I}_1 + \phi^2 \hat{I}_0. \] (18)

Below we shall prove

**Theorem 1** For \( n \geq 3 \), there are polynomials \( \psi_n = \psi_n(\phi, \phi', ..., \phi^{(n-1)}) \) of degree \( n - 2 \), such that

\[ I_n = \sum_{k=0}^{n} \binom{n}{k} (\phi^k - \psi_k) \hat{I}_{n-k}, \quad n = 0, 1, 2, .... \] (19)

By definition \( \psi_0 = \psi_1 = \psi_2 = 0 \).

The properties (ii) and (iii) hold in the general case:

**Theorem 2** \( \{I_n, I_m\} = 0 \) for all \( n, m \geq 0 \), and \( DI_n = nI_{n-1} \) for all \( n \geq 0 \).
We shall also make use of the following result. Needless to say, it holds in the sense of equivalence of functionals.

**Theorem 3** For any \( f \in C^\infty(\mathbb{C}^{n+1}) \) we have

\[
D (f(I_0, I_1, ..., I_n)) = \sum_{\nu=0}^{n} \frac{\partial f}{\partial I_\nu}(I_0, I_1, ..., I_n)\nu I_{\nu-1}.
\] (20)

This leads to a bundle where the 'quantum space' \( \mathbb{C}[I_0, I_1, I_2, ...] \) of all polynomials in the variables (conservation laws) \( I_n \), projects down to the 'classical space' \( \mathbb{C}[I_1/I_0] \). The fibre is the kernel of \( D \) and may be identified with classical semi-invariants. We refer to the precise details in §5 below.

## 2 Preliminaries and background.

### 2.1 Symmetries and conservation laws for linear heat equations.

Assume that \( u \) and \( v \) satisfy

\[
\dot{u} + \frac{1}{2}u'' - Vu = 0; \quad -\dot{v} + \frac{1}{2}v'' - Vv = 0,
\] (21)

where \( V \) is a given, smooth, potential. Then

\[
\frac{d}{dt} \int uv \, dq = 0,
\] (22)

assuming sufficiently rapid decrease at infinity of \( u \) and \( v \). Let \( f = f(t, q) \) and put \( Kf = f + \frac{1}{2}f'' - Vf \). It is easy to show (see Brandão and Kolsrud [4]) that

\[
\frac{d}{dt} \int f uv \, dq = \int \partial f uv \, dq = \int \partial^* f uv \, dq,
\] (23)

where

\[
\partial f : = \frac{1}{u}K(fu) = \dot{f} + \frac{1}{2}f'' + \frac{u'}{u}f,
\] (24)

\[
\partial^* f : = -\frac{1}{v}K^\dagger(fv) = \dot{f} - \frac{1}{2}f'' - \frac{v'}{v}f.
\] (25)
The first identity can be written
\[ \partial = u^{-1}Ku \quad \text{or} \quad u\partial u^{-1} = K. \] (26)
Suppose the linear PDO \( \Lambda = T \partial_t + Q \partial_q + U \) belongs to the heat Lie algebra of \( K \):
\[ [K, \Lambda] = K\Lambda - \Lambda K = \Phi \cdot K \] (27)
for some function \( \Phi = \Phi_\Lambda \). Using \( u\partial u^{-1} = K \) we obtain
\[ u\partial (\Lambda u/u) = K\Lambda u = (\Phi_\Lambda + \Lambda)Ku = 0. \] (28)
This is an alternative way of expressing that \( \Lambda u \cdot v \) is the density of a conservation law:
\[ \frac{d}{dt} \int \Lambda u \cdot v \, dq = \int \partial(u^{-1}\Lambda u) \cdot uv \, dq = \int (\Phi_\Lambda + \Lambda)Ku \cdot uv \, dq = 0. \] (29)
In more detail, the equation \( \partial(\Lambda u/u) = 0 \) above may be written
\[ \partial \left( T\frac{\dot{u}}{u} + Q\frac{\dot{u}'}{u} + U \right) = 0, \] (30)
very much as in the classical case, where the Noether theorem leads to a constant of motion of the form \( ET + Tp + U \), where \( E = -\frac{1}{2}p^2 + V \) (Euclidean convention) and \( p = \dot{q} \). The coefficients \( \dot{u}u^{-1} = -\frac{1}{2}u''u^{-1} + V \) and \( \dot{u}'u^{-1} \) are, respectively, the energy density and the momentum density in a form that emphasises the backward motion, and the classical total time derivative is replaced by \( \partial \).

The basic density is the ground state \( I_0 = uv \), and for instance \( u'v = \frac{\dot{u}}{u} \cdot I_0 \) is an equivalent form for \( \frac{1}{2}(u_1v - uv_1) = \dot{I}_1(0) \). We write \( \bar{p} \) for the moment density \( u' \). Then we have the Euclidean Newton/Hamilton equations (cf Landau-Lifshitz [20], Eq. (19.3))
\[ \partial q = \bar{p} = -H\bar{p}, \quad \partial p = V' = Hq, \] (31)
with \( H \) denoting the Euclidean Hamiltonian \(-\frac{1}{2}p^2 + V \). Similarly the energy density \( \bar{E} \) satisfies \( \partial \bar{E} = \dot{V} \).

Repeating the argument above one finds that for \( \Lambda_j \) in the Lie algebra of \( K \) and \( s_j \geq 0 \) integers, we have
\[ \frac{d}{dt} \int \Lambda_1^{s_1} \cdots \Lambda_m^{s_m} u \cdot v \, dq = 0, \] (32)
i.e. \( \Lambda_1^{s_1} \cdots \Lambda_m^{s_m} u \cdot v \) is the density of a conservation law. Its probabilistic version is that \( \Lambda_1^{s_1} \cdots \Lambda_m^{s_m} u/u \) is a martingale for a certain diffusion process.

7
2.2 The operator $D$.

Consider a slightly more general situation, in which our space is built from one independent variable $q$ and $m$ dependent variables $u^\alpha$, $\alpha = 1, \ldots, m$. We define $\mathcal{A}$ as all smooth ($C^\infty$) functions of the variables $q$ and $u^\alpha_j$, where $j = 0, \ldots, n$, so that the order $n$ is arbitrary but finite.

Given a (canonical) vector field

$$X = a^\alpha \frac{\partial}{\partial u^\alpha}, \quad a^\alpha \in \mathcal{A}$$

(33)

together with its (infinite order) extension (prolongation), Ibragimov [10-12] or Olver [26],

$$\tilde{X} = \sum_{k \geq 0} a_k^\alpha \frac{\partial}{\partial u_k^\alpha}, \quad a_k^\alpha = D^k a^\alpha.$$  

(34)

(Any vector field as above lifts in a canonical way to a vector field on the appropriate jet bundle.) Summation over the repeated index $\alpha = 1, \ldots, m$ is understood.

Write $\Xi$ for the variational counterpart of $X$:

$$\Xi = a^\alpha \delta \frac{\partial}{\partial u^\alpha}. \quad \text{(35)}$$

Lemma 1 For $I \in \mathcal{A}$ we have

$$(\tilde{X} - \Xi)I = D \sum_{k \geq 0} \hat{a}_k^\alpha \frac{\partial I}{\partial u_k^\alpha},$$

where $\hat{a}_k^\alpha$ denotes the differential operator

$$\hat{a}_k^\alpha := D^k a^\alpha - D^{k-1} a^\alpha D + \ldots + a^\alpha (-D)^k.$$

Proof: Fix $I$. It suffices to consider the case when $I$ only depends on one variable $u$ and its derivatives. Write $\partial_j I = \partial I/\partial u_j$. Then

$$(\tilde{X} - \Xi)I = \sum_{k=0}^{\infty} (D^k a \partial_k I - a(-D)^k \partial_k I) = (Da \partial_1 I + aD \partial_1 I)$$

$$+ (D^2 a \partial_2 I - aD^2 \partial_2 I) + \ldots + (D^n a \partial_n I + (-1)^{n-1} D^n \partial_n I) + \ldots$$

$$= D(a \partial_1 I + (Da \partial_2 I - aD \partial_2 I)) + (D^2 a \partial_3 I - DaD \partial_3 I + aD^2 \partial_3 I) + \ldots.$$
as stated.

We define the operator

$$D_X := D^{-1}(\Xi - \tilde{X}).$$  \hfill (36)

The following is a key result:

**Theorem 4** On the space of equivalence classes of functionals $\mathcal{A} = \mathcal{A}/D\mathcal{A}$, we have

$$D_X = -\sum_{k \geq 0} (k + 1)a^\alpha_k \frac{\partial}{\partial u_{k+1}^\alpha}.$$  \hfill (37)

In particular, $D_X$ is a derivation on $\mathcal{A}$.

**Proof:** Clearly

$$(D^k a^\alpha - D^{k-1} a^\alpha D + ... + a^\alpha (-D)^k) \frac{\partial I}{\partial u_{k+1}^\alpha}$$

is equivalent to

$$(k + 1)D^k a^\alpha \frac{\partial I}{\partial u_{k+1}^\alpha}$$

modulo the image of $D$.

**Corollary 1** If $\tilde{X} I = 0$, then

$$D_X I = D^{-1} \Xi I = D^{-1} a^\alpha \frac{\delta I}{\delta u^\alpha}. \hfill (38)$$

In particular, for $X = v\partial/\partial v - u\partial/\partial u$ and $I_0 = uv$, we have

$$D_X I = D^{-1} \left( v \frac{\delta I}{\delta v} - u \frac{\delta I}{\delta u} \right) = DI = \{-qI_0, I\}, \hfill (39)$$

provided $\tilde{X} I = 0$.

**Definition:** Let $m = 2$. A differential function on the form constant times

$$u_0^{s_0} u_1^{s_1} \cdots u_N^{s_N} v_0^{t_0} v_1^{t_1} \cdots v_N^{t_N},$$

where $s_j, t_k \in \mathbb{N}$, is a *balanced monomial* if the number of $u_j$s and $v_k$s are equal, i.e., if $\sum s_j = \sum t_k$. A finite sum of balanced monomials is a *balanced polynomial*. 
Corollary 2 Let \( X = v \partial/\partial v - u \partial/\partial u \). Any functional of the form \( I = f(A_1, \ldots, A_N) \), where \( f \) is \( C^\infty \) and \( A_j \) are balanced polynomials, satisfies \( \tilde{X}I = 0 \). Consequently, 

\[
DI = \sum_{k \geq 0} (k + 1) \left( u_{k+1} \frac{\partial I}{\partial u_k} - v_{k+1} \frac{\partial I}{\partial v_k} \right)
\]

for such \( I \).

**Proof.** It suffices to show that \( \tilde{X}F = 0 \) for any balanced monomial \( F \). Thus, consider \( F = u_{s_0} u_{t_1} \ldots u_N v_{t_0} v_{t_1} \ldots v_t \) with \( \sum s_n = \sum t_n \). Then

\[
\tilde{X}F := \sum_n \left( v_n \frac{\partial F}{\partial v_n} - u_n \frac{\partial F}{\partial u_n} \right) = \sum (t_n - s_n) F = 0.
\]

**Remark:** All elements in the NLS hierarchy are balanced monomials.

3 Theorem 1.

3.1 Proof of Theorem 1.

We start with the case \( \lambda = 0 \), and consider a bit more generally, a sequence of functionals

\[
T_n = \sum_{k=0}^{n} \binom{n}{k} \alpha_k \hat{I}_{n-k}(0), \quad n = 0, 1, 2, \ldots,
\]

where \( \alpha_0 \equiv 1, \alpha_1, \alpha_2, \ldots \) are given, smooth, functions, and, as above, \( \hat{I}_s(0) = \frac{1}{2}(u_1 v + (-1)^s u v_s), s = 0, 1, 2, 3, \ldots \)

We get

\[
\frac{\delta T_n}{\delta v} = u_n + \frac{1}{2} \sum_{k=1}^{n} \binom{n}{k} \left( \alpha_k u_{n-k} + D^{n-k} \alpha_k u \right),
\]
which, upon resumming, becomes

\[ u_n + n\alpha_1 u_{n-1} + \frac{1}{2} \sum_{k=2}^{n} \left( 2 \binom{n}{k} \alpha_k + \sum_{j=1}^{k-1} \binom{n}{k-j} \binom{n}{j} D^j \alpha_{k-j} \right) \]

Choosing \( \alpha_k =: \phi^k = \phi^k - \psi_k \),

\( T_n \) becomes \( I_n(0) \). Assume that \( \psi_3, ..., \psi_{n-1} \) have been chosen so that

\[ \frac{\delta I_n(0)}{\delta v} = u_n + \sum_{k=1}^{n-1} \Lambda^{k-1} \phi \cdot u_{n-k} + A_n u, \]

where

\[ A_n = \phi^n - \psi_n + \frac{1}{2} \sum_{j=1}^{n-1} \binom{n}{j} D^j (\phi^{n-j} - \psi_{n-j}) = A_0^n - \psi_n. \]

Our task is to choose \( \psi_n \), satisfying the criteria in Theorem 1, such that \( \phi^n - \psi_n = \Lambda^{n-1} \phi \). Before doing this, we remark that the coefficient

\[ \frac{1}{2} \binom{n}{k} \left( 2\alpha_k + \sum_{j=1}^{k-1} \binom{k}{j} D^j \alpha_{k-j} \right) \]

of \( u_{n-k} \) in the formula above only depends on \( n \) through the binomial coefficient \( \binom{n}{k} \). It shows that : \( \phi^k : \), the renormalisation of the \( k \)th power of \( \phi \), is independent of \( n \).

We note that both \( A_0^n \) and \( B_n \equiv \Lambda^{n-1} \phi \) are polynomials satisfying the criteria in Theorem 1. The choice \( \psi_n = A_0^n - B_n \) determines \( \psi_n \) uniquely.

In the general case, \( I_n = \hat{I}_n(0) + J_{n-2} \), where \( J_{n-2} \) only depends on \( u_j, v_j, : \phi^j : \) for \( j \leq n-2 \). The renormalisation term : \( \phi^n : \) only occurs in the first term \( \hat{I}_n(0) \), which we have already discussed. A similar argument works for \( \delta I_n/\delta u \). Theorem 1 follows.
3.2 Comments on renormalisation.

We consider the linear case $\lambda = 0$. Variation of the space-time action $\int \int L \, dtdq$, with (space-time) Lagrangian density

$$L = \frac{1}{2}(uv - \dot{uv}) + \frac{1}{2}l_2,$$

(41)

leads to the Euler-Lagrange equations $\dot{u} = -\frac{1}{2} \frac{\delta I_2}{\delta v}$, $\dot{v} = \frac{1}{2} \frac{\delta I_2}{\delta u}$. (See Ibragimov and Kolsrud [13].) In more detail:

$$\dot{u} + \frac{1}{2} u_2 + \phi u_1 + \frac{1}{2}(\phi'^2 + \phi')u = \dot{u} - Hu = 0;$$

(42)

$$-\dot{v} + \frac{1}{2} v_2 - \phi v_1 + \frac{1}{2}(\phi'^2 - \phi')v = -\dot{v} - H^1 v = 0.$$

(43)

Remark: The variational principle appears for the Schrödinger equation in Goldstein [8], and the same trick is used in classical mechanics in Morse and Feshbach [25]. It is related to the Hilbert integral, in, e.g. [1].

The Hamiltonian $H$ is evidently unsymmetric, and $H \to H^1$ precisely when $\phi \to -\phi$. In the non-linear case $\lambda \neq 0$ the equations become

$$\dot{u} - Hu = \lambda u^2 v; \quad -\dot{v} - H^1 v = \lambda u v^2.$$

(44)

Writing $:\phi^n : \equiv \phi^n - \psi_n$, as above, one finds

$$:\phi^3 : = \phi^3 - \frac{1}{2}\phi''; \quad :\phi^4 : = \phi^4 - 2\phi\phi'' - 3\phi'^2,$$

(45)

$$:\phi^5 : = \phi^5 - 5\phi^2\phi'' - 15\phi\phi'^2,$$

(46)

$$:\phi^6 : = \phi^6 - 40\phi^3\phi'' - 45\phi^2\phi'^2 + 6\phi\phi^{iv} + 30\phi\phi'' + 10\phi''^2, \ldots.$$

(47)

Examples: It is known that the Lie algebra is maximal when the potential is quadratic in $q$. Then the Lie algebra contains an $sl_2$, in addition to the Heisenberg algebra. The latter occurs only in this case, and the case of inverse square potential, $V = c/q^2$.

(i) In the case $\phi = cq$ no renormalisation occurs for $n = 3$. For $c = 1$ we get

$$:q^n : = q^n - 3\left(\frac{n}{4}\right)q^{n-4}, \quad n \geq 4.$$

(48)

This is related to the harmonic oscillator and the $:q^n :$ share some properties with Hermite polynomials, notably $D : q^n : = n : q^{n-1} :$. However, to
get a potential with correct sign we must take \( c \) imaginary. (For Gaussian renormalisation, see Simon [29].)

(ii) The choice \( \phi = c/q \) is related to the inverse square potential. For \( c = 1 \) we have

\[
: q^{-n} := \lambda_n q^{-n}, \quad n \geq 0. \tag{49}
\]

The first few odd-indexed : \( q^{-n} \): vanish. In particular, : \( q^{-3} := 0 \). This happens only in this case. Here, one can keep \( c \) real and get correct sign for one, but not both, of the potentials, \( \frac{1}{2}(\phi^2 \pm \phi') \). Many formulæ become particularly simple in this case, because \( \Lambda(1/q) = 0 \).

Remark: Incidentally, \( u = c_1/q, \ v = c_2/q \) solves the stationary NLS equation \( u'' - 2u^2v = 0, \ v'' - 2uv^2 = 0 \), provided \( c_1c_2 = 1 \). The same holds for the stationary KdV system obtained from \( \hat{I}_3 \), viz. \( u''' - 6uu'v = 0, \ v''' - 6uvv' = 0 \), and possibly for corresponding higher order equations. Cf. Treves [32].

4 Proof of Theorem 2.

4.1 Involutivity.

We first show that \( C \) is symmetric w.r.t. the bracket.

Lemma 2 Assume two functionals \( F \) and \( G \) commute with \( I_0 \). Then

\[
\{CF, G\} = - \{CG, F\} = \{F, CG\}. \tag{50}
\]

Proof. The second identity is a consequence of the bracket being antisymmetric. To prove the first identity, we write \( \{CF, G\} = \int \left( \left( \Lambda^1 \frac{\delta F}{\delta u} - 2\lambda vD \frac{\delta G}{\delta v} - \Lambda \frac{\delta G}{\delta u} - 2\lambda uD \frac{\delta F}{\delta v} \right) \frac{\delta G}{\delta u} - \Lambda \frac{\delta F}{\delta v} - 2\lambda D \frac{\delta F}{\delta u} \right) dq \right. \]

\[
= \int \left( \left( \frac{\delta F}{\delta u} \Lambda \frac{\delta G}{\delta v} - \frac{\delta F}{\delta v} \Lambda \frac{\delta G}{\delta u} - 2\lambda D \left( \frac{\delta G}{\delta v} - \frac{\delta F}{\delta u} \right) \right) \frac{\delta G}{\delta u} - \Lambda \frac{\delta F}{\delta v} - 2\lambda D \frac{\delta F}{\delta u} \right) dq \]

\[
= \int \left( \frac{\delta F}{\delta u} \Lambda \frac{\delta G}{\delta v} - \frac{\delta F}{\delta v} \Lambda \frac{\delta G}{\delta u} - 2\lambda D \frac{\delta F}{\delta u} \right) dq,
\]

where we have used the definition \( D = D^{-1}(v\delta/\delta v - u\delta/\delta u) \). Performing a partial integration in the last term, we clearly get \( -\{CG, F\} \). This proves our claim.
To prove that all $I_n$ commute, we assume that
\[
\{I_j, I_k\} = 0, \quad 0 \leq j, k \leq n,
\]
and prove that it can be extended to the first $n + 1$ conservation laws. If $j < n$, the above observation shows
\[
\{I_{n+1}, I_j\} = \{CI_n, I_j\} = \{I_n, I_{j+1}\},
\]
which vanishes by hypothesis. The identity also shows that
\[
\{I_{n+1}, I_n\} = \{I_n, I_{n+1}\},
\]
which since the bracket is anti-symmetric allows us to conclude that, indeed also $\{I_{n+1}, I_n\} = 0$.

4.2 Proof of $DI_n = nI_{n-1}$.

It is clear from the results in §2.2 that for any balanced $I$ we have $D(f(q)I) = f(q)DI$. Hence the formula in Theorem 1 leads to
\[
DI_n = \sum_{k=0}^{n} \binom{n}{k} :\phi^k: DI_{n-k} = \sum_{k=0}^{n} \binom{n}{k} :\phi^k: (n-k)\hat{I}_{n-k-1} = nI_{n-1},
\]
provided the $\hat{I}_n$ from the NLS-hierarchy fulfill (ii). This is what we shall prove. We drop the hats from now on, and assume that $\lambda = 1$.

Write $a_n := DI_n$. In general,
\[
a_n = D^{-1} \left( \frac{\delta I_n}{\delta v} - u \frac{\delta I_n}{\delta u} \right) = \int -q \left( \frac{\delta I_n}{\delta v} - u \frac{\delta I_n}{\delta u} \right) dq = \int -qa'_n dq.
\]

Use of the creation operator $C$ and partial integration leads to the relation
\[
a'_{n+1} = u^{(n+1)}v + (-1)^n uv^{(n+1)}
\]
\[\quad -2 \left( (a_1 u)^{(n-1)}v + (-1)^n u(a_1 v)^{(n-1)} \right) + \ldots \]
\[\quad + (a_{n-2} u)^{(n-1)}v - u(a_{n-2} v)^{(n-1)} + (a_{n-1} u)'v + u(a_{n-1} v)'
\]
\[= A_{n+1} + A_{n-1} + \ldots \]

where the index on the right refers to the total number of derivatives.
Assume now that $a_k = kI_{k-1}$ for all $k \leq n$. The terms of lowest order will come from

$$-2((a_{n-1}u)'v + u(a_{n-1}v)')$$

if $n$ is even, and from

$$-2((a_{n-2}u)''v - u(a_{n-2}v)'' + (a_{n-1}u)'v + u(a_{n-1}v)')$$

if $n$ is odd.

In the former case, the hypothesis yields

$$a_{n+1}' = -2(n - 1)((I_{n-2}uv)' + I'_{n-2}uv) + \text{higher order terms}.$$ 

In general,

$$I_{2m} = c_{2m}(uv)^{m+1} + \text{higher order terms},$$

where the coefficient is (if $(-1)!! = 1$)

$$c_{2m} = (-2)^m \frac{(2m - 1)!!}{(m + 1)!}, \quad m = 0, 1, 2, \ldots$$

This expression can be found using the following formulae for $C^2$:

$$\frac{\delta I_{k+2}}{\delta u} = D^2 \frac{\delta I_k}{\delta u} + 2(a_k v)' - 2a_{k+1} v,$$

$$\frac{\delta I_{k+2}}{\delta v} = D^2 \frac{\delta I_k}{\delta v} - 2(a_k u)' - 2a_{k+1} u.$$

With $n = 2m$, and writing $s := uv$, the terms of lowest order are

$$-2(2m - 1)c_{2(m-1)}((s^{m+1})' + (s^m)'s)$$

$$= -2(2m - 1)c_{2(m-1)}(2m + 1)s^m s'$$

$$= (2m + 1)(-2)^m \frac{(2m - 1)!!}{m!} \frac{(s^{m+1})'}{m + 1} = (2m + 1)(c_{2m}s^{m+1})',$$

which proves the assertion in this case.

In the case when $n$ is odd, $n = 2m + 1$, the lowest order terms for $a'_{2m+2}$ are obtained from

$$-2(2m(I_{2m-1} s)')$$

$$+ 2m I'_{2m-1} s + (2m - 1)I_{2m-2} a$$

$$+ (2m - 1)(I_{2m-2} a)'$$,
where, in addition to \( s = uv \), we have written \( a := u'v - uv' \). In general, 
\[ I_{2m+1} = c_{2m+1}s^ma + \text{ h. o. t.} \]
for some constant \( c_{2m+1} \). Hence the middle terms above are
\[
2mc_{2m-1}(s^{m-1}a)'s + (2m - 1)c_{2m-2}(s^m)'a
= (2mc_{2m-1}(m - 1) + (2m - 1)mc_{2m-2})s^{m-1}s'a + 2mc_{2m-1}s^ma'
= (2(m - 1)c_{2m-1} + (2m - 1)c_{2m-2})(s^m)'a + 2mc_{2m-1}s^ma'
= 2mc_{2m-1}(s^m)'
\]
provided \( 2(m - 1)c_{2m-1} + (2m - 1)c_{2m-2} = 2mc_{2m-1} \), i.e.
\[ c_{2m-1} = \frac{2m - 1}{2}c_{2(m-1)}. \]
One may deduce this formula from the formula for \( c_{2m} \) together with theformulae for \( C^2 \) displayed above.

The lowest order terms become
\[-2(2\cdot 2mc_{2m-1} + (2m - 1)c_{2m-2})(s^m)'.\]
The coefficient can be written
\[-2(2m - 1)(2m + 1)c_{2m-2} = 2(m + 1)\cdot \frac{2m + 1}{2}(-2)^m\frac{(2m - 1)!!}{(m + 1)!} = 2(m + 1)c_{2m+1},\]
which proves our claim
\[ a_{2(m+1)}' = 2(m + 1)c_{2m+1}(s^m)' + \text{ h. o. t.} \]

By induction, we may assume that all terms of order strictly less than the highest order, viz. \( n + 1 \), satisfy the corresponding identity. It remains to prove that \( J_n := \frac{1}{2}(u^{(n)}v + (-1)^n uv^{(n)}) \) fulfill
\[ DJ_n = nJ_{n-1} \quad \text{for all } n. \]
This is the relation \( DI_n = nI_{n-1} \) in the free case. It follows immediately from noting that
\[ u^{(n)}v - (-1)^n uv^{(n)} = D \left( u^{(n-1)}v - u^{(n-2)}v' + \cdots + (-1)^{n-1}uv^{(n-1)} \right), \]
with each term within the parentheses being equivalent to \( I_{n-1} \).
5 \( \mathbb{C}[I_0, I_1, ...] \) and D-invariant polynomials.

5.1 General setting.

Suppose we are given

\[ I_0, I_1, I_2, ..... \]  \hspace{2cm} (52)

and a derivation \( D \) for which

\[ DI_n = nI_{n-1}, \quad n \geq 0. \]  \hspace{2cm} (53)

Replacing \( I_n \) by \( I_n := I_n/I_0 \), one finds that once again, \( DI_n = nI_{n-1} \), and \( I_0 = 1 \). We may therefore assume \( I_0 = 1 \).

Each \( I_n \) is assigned the degree \( n \). We may then define the degree of a monomial

\[ M_\alpha = I_1^{\alpha_1}I_2^{\alpha_2} \cdots I_s^{\alpha_s}, \quad \alpha_i \in \mathbb{N}, \]  \hspace{2cm} (54)

as

\[ \deg M_\alpha := \sum_{j=1}^{s} j\alpha_j = ||\alpha||. \]  \hspace{2cm} (55)

A linear combination of monomials of the same degree,

\[ P = \sum_{||\alpha||=N} c_\alpha M_\alpha, \]  \hspace{2cm} (56)

is a homogeneous polynomial of degree \( N \). General polynomials in \( I_1, I_2, .... \) are linear combinations of homogeneous polynomials. This way, the ring of polynomials \( \mathbb{P} := \mathbb{C}[I_1, I_2, ....] \) (the field of scalars does not seem so important) gets a natural grading by the degree:

\[ \mathbb{P} = \bigoplus_{N=0}^{\infty} \mathbb{P}_N, \]  \hspace{2cm} (57)

where \( \mathbb{P}_N \) denotes all homogeneous polynomials of degree \( N \).

\( D \) being a derivation, we have

\[ D\{f(I_1, I_2, ..., I_n)\} = \sum_{j=1}^{n} \frac{\partial f}{\partial I_j}(I_1, I_2, ..., I_n)DI_j \]  \hspace{2cm} (58)

for every polynomial \( f \).
Define polynomials $K_N$, $N \geq 2$, by the formula

$$K_N = \sum_{\nu=0}^{N-2} (-1)^\nu \binom{N}{\nu} I_1^\nu I_{N-\nu} + (-1)^{N-1}(N-1)I_1^N.$$  \hfill (59)

Then $D$ annihilates all $K_N$:

$$DK_N = 0, \quad N \geq 2,$$  \hfill (60)

as one easily checks. Together, all the $K_N$ form an algebra of invariant polynomials. (Since $D$ is a derivation, it annihilates the whole algebra, according to the Leibniz rule.)

**Theorem 5** \( P_N \) has the decomposition

$$P_N = H_N \oplus V_N,$$  \hfill (61)

where

$$H_N := I_1P_{N-1} \quad \text{and} \quad V_N = \mathbb{C}[K_2, K_3, ..., K_N] \cap P_N.$$  \hfill (62)

**Proof:** The monomials in $P_N$ are divided into two classes according to whether the monomial contains a factor $I_1$ or not. Clearly, the first case corresponds to the space $H_N$, as defined above.

The remaining monomials can be identified with those partitions of $N$, for which all summands fulfil $2 \leq \lambda_i \leq N$. The trivial partition, i.e. $N$ itself, is identified with $K_N$. The other partitions satisfy $2 \leq \lambda_i \leq N - 2$, and can be identified with $K_{\lambda_1}K_{\lambda_2} \cdots K_{\lambda_s}$. (Typically there are repetitions of the $K_j$.) This proves the theorem.

The theorem leads to

$$P_N/V_N \simeq H_N.$$  \hfill (63)

From $K_2 = 0$ follows $I_2 = I_1^2$ and $K_3 = 0$ leads to $I_3 = 3I_1I_2 - 2I_1^3 = I_1^3$ etc. We get $I_n = I_1^n$ for $0 \leq n \leq N$. In particular, $H_N$ can be identified with the one-dimensional space that $I_1^N$ generates.

The result as $N \to \infty$ may be written

**Corollary 3** \( \mathbb{C}[I_1, I_2, ....]/\mathbb{C}[K_2, K_3, ....] = \mathbb{C}[I_1] \).

The invariant polynomials have integer coefficients and it holds that \( \mathbb{Z}[I_1, I_2, ....]/\mathbb{Z}[K_2, K_3, ....] = \mathbb{Z}[I_1] \).
5.2 Invariants and semi-invariants.

To see how the $K_N$ arise, we start from the quotient $I_2/I_1^2$, homogeneous of degree zero. Acting on it by the vector field $I_1 D$ we get a new function with the same homogeneity. The obtained relation may be written as

$$-I_1 D (I_2/I_1^2) = 2K_2/I_1^2. \quad (64)$$

More generally, one finds

$$-I_1 D \left( \frac{I_{n+1}}{I_1^{n+1}} \right) = (n+1) \sum_{j=0}^{n-1} \binom{n}{j} \frac{K_{n+1-j}}{I_1^{n+1-j}}, \quad (65)$$

This can be used to obtain a kind of generating function relation between the $I_n$ and the $K_N$.

If we bring back $I_0$, the first few $K_N$ are

$$K_2 = I_0 I_2 - I_1^2, \quad K_3 = I_0^2 I_3 - 3I_0 I_1 I_2 + 2I_1^3, \quad (66)$$
$$K_4 = I_0^3 I_4 - 4I_0^2 I_1 I_3 + 6I_0 I_1^2 I_2 - 3I_1^4 \quad (67)$$

These expressions are semi-invariants, or relative invariants, related to forms, i.e. homogeneous polynomials in two variables, and projective representations of $GL(2, \mathbb{R})$ or $SL(2, \mathbb{R})$. We refer to Gurevich [9], Ibragimov [10], and Olver [27, 28] for more about this classical, fascinating subject.

$K_2$, the discriminant, is a true invariant. It can be written $\begin{vmatrix} I_0 & I_1 \\ I_1 & I_2 \end{vmatrix}$.

Instead of $K_4$ we could have chosen the well-known invariant (for quartic polynomials) $\tilde{K}_4 := I_0 I_4 - 4I_1 I_3 + 3I_2^2$. Except for the (irrelevant) factor $I_0^2$, $K_4$ and $\tilde{K}_4$ differ by a multiple of $K_2^2$.

The $3 \times 3$ determinant with rows, from the top, $(I_0 I_1 I_2)$, $(I_1 I_2 I_3)$, $(I_2 I_3 I_4)$, is another well-known invariant related to quartic polynomials (Olver, [27], p. 97). It is a linear combination of $K_2^3$, $K_3^2$ and $K_2 K_4$.

For $N$ even, there is an alternative choice of $K_N$, , with $\tilde{K}_4$ as a special case, namely

$$\tilde{K}_{2n} := \sum_{k=0}^{n-1} (-1)^k \binom{2n}{k} I_k I_{2n-k} + (-1)^n \binom{2n}{n} I_n^2.$$
5.3 Proof of Theorem 3.

Every $I_n$ is a balanced polynomial with $q$-dependent coefficients. Hence the formula in Corollary 2 applies. It shows that $D$ is a derivation on functionals of the form $f(I_0, \ldots, I_n)$. Thus

$$D\{f(I_0, \ldots, I_n)\} = \sum_{\nu=0}^{n} \partial_\nu f(I_0, \ldots, I_n) DI_\nu,$$

which according to Theorem 2 yields the desired result.

**Corollary 4** Let $(I_n)_{0}^{\infty}$ be given by Theorem 1. Then

$$\mathbb{C}[I_0, I_1, I_2, \ldots]/\mathbb{C}[K_2, K_3, \ldots] = \mathbb{C}[I_0, I_1, \ldots]/\ker D = \mathbb{C}[I_1/I_0] \simeq \mathbb{C}[p].$$

5.4 A final remark.

One may ask what happens if we switch between $I_1$ and $I_1^* := -qI_0$. The sequence $I_n^* := -q^n I_0$ forms, of course, an abelian algebra. Using $D^* := ad_{I_1}$, we get the same derivation property as above. In this case, however, the recursion operator $I_n^* \rightarrow I_{n+1}^*$ is trivial.

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