Position Dependent NLS Hierarchies: Involutivity, Commutation Relations, Renormalisation and Classical Invariants

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May – October 2005

Abstract

We consider a family of explicitly position dependent hierarchies $(I_n)_0^\infty$, containing the NLS (non-linear Schrödinger) hierarchy. All $(I_n)_0^\infty$ are involutive and fulfill $\mathsf{D}I_n = nI_{n-1}$, where $\mathsf{D} = D^{-1}V_0$, V_0 being the Hamiltonian vector field $v\frac{\delta}{\delta v} - u\frac{\delta}{\delta u}$ afforded by the common ground state $I_0 = uv$. The construction requires renormalisation of certain function parameters.

It is shown that the 'quantum space' $\mathbb{C}[I_0, I_1, ...]$ projects down to its classical counterpart $\mathbb{C}[p]$, with $p = I_1/I_0$, the momentum density. The quotient is the kernel of D. It is identified with classical semi-invariants for forms in two variables.

Introduction: Consider in 1+1 dimensions the (free) heat equation system (u and v are functions of time, t, and space q)

$$\dot{u} + \frac{1}{2}u'' = 0; \qquad -\dot{v} + \frac{1}{2}v'' = 0. \tag{1}$$

With appropriate 'boundary conditions' on u and v (e.g. rapid decrease at infinity or periodicity), all $I_n := \frac{1}{2}(u^{(n)}v + (-1)^n uv^{(n)})$ are *conservation laws*:

$$\frac{d}{dt}\int I_n \, dq = 0, \quad n = 0, 1, 2, ... \tag{2}$$

This is an immediate consequence of the equations being invariant under space translations. There is an additional first order conservation law, viz. $tI_1 - qI_0$.

The counterpart of $(I_n)_0^\infty$ for the free classical (Newton) equation $\ddot{q} = 0$ is the sequence $(p^n)_0^\infty$, of *constants of motion.* ($p = \dot{q}$, as usual.) Obviously, all p^n commute in the Poisson bracket

$$\{\xi, \eta\} := \frac{\partial \xi}{\partial p}\frac{\partial \eta}{\partial q} - \frac{\partial \xi}{\partial q}\frac{\partial \eta}{\partial p} \tag{3}$$

The additional first order (in p) constant of motion $pt - q$ satisfies

$$\{pt - q, p^n\} = np^{n-1} = dp^n/dp. \tag{4}$$

(t is looked upon as a parameter.)

Similarly, with the (field theory) bracket $\{F, G\} := \int \left(\frac{\delta F}{\delta u}\frac{\delta G}{\delta v} - \frac{\delta F}{\delta v}\frac{\delta G}{\delta u}\right) dq$, we have

$$\{tI_1 - qI_0, I_n\} = \{-qI_0, I_n\} =: \mathsf{D}I_n = nI_{n-1}. \tag{5}$$

This is of course related to $\langle 1, p, pt - q\rangle$ and $\langle I_0, I_1, tI_1 - qI_0\rangle$, respectively, being representations of the Heisenberg algebra.

Suppose now that we form $\mathbb{C}[I_0, I_1, I_2, ...]$: all polynomials in the variables $I_0, I_1, I_2, ...$ What, if any, is the relation to the classical version, viz. $\mathbb{C}[p]$, all polynomials in p?

Below it is shown that there is a projection

$$\mathbb{C}[I_0, I_1, I_2, ...] \to \mathbb{C}[I_1/I_0] \simeq \mathbb{C}[p] \tag{6}$$

with 'fibre' $\ker \mathsf{D}$, which in its turn is related to the classical 19th century semi-invariants of Cayley and others. See Gurevich [9], Ibragimov [10], Olver [27, 28].

The paper is devoted to this and some related questions, among them renormalisation, for a wider class of commuting conservation laws, containing a version of the non-linear Schrödinger hierarchy, NLS.

As background serve the papers on invariance properties, including behaviour under mappings between manifolds, for Schrödinger and related diffusion processes [4, 6, 14, 16, 17, 18, 22, 30, 31], in particular the case of Gaussian diffusions [2, 3, 18]. At the centre of much of this is the heat Lie algebra, first described by Lie in 1881 [21]. See e.g. Ibragimov [10, 11, 12] and Olver [28]. Other general background references are [2], [6] and [19, 20], and for the NLS equation primarily [7], together with [23, 24] and [32].

1 Outline and formulation of results

Consider all C^∞ curves $(u(q), v(q))$, $q \in \mathbb{R}$, in $\mathbb{C}^2$. We are interested in *functionals* or *differential functions* of the form $F = f(q, u, u_1, ..., u_n; v, v_1, ..., v_n)$, where $u_j = u^{(j)}$, $v_j = v^{(j)}$ and where it is understood that all the u_i and v_j depend on q, the coordinate in the base space. Here f is C^∞ in the appropriate space, a jet bundle. With $D = d/dq$ we form variational derivatives:

$$\frac{\delta F}{\delta u} = \frac{\partial F}{\partial u} - D\frac{\partial F}{\partial u_1} + D^2\frac{\partial F}{\partial u_2} + ..., \quad \frac{\delta F}{\delta v} = \frac{\partial F}{\partial v} - D\frac{\partial F}{\partial v_1} + D^2\frac{\partial F}{\partial v_2} + ... \tag{7}$$

The variational gradient δF of F is the transpose of the vector $(\delta F/\delta u, \delta F/\delta v)$. Two functionals F and G are identified whenever $\delta(F-G) = 0$. This is equivalent to saying that $F - G \in \operatorname{im} D$. The interpretation is that we have put extra 'gauge' conditions on u and v, e.g. on their behaviour at infinity.

The bracket is, when emphasising the *Hamiltonian densities* F and G,

$$\{F, G\} := \frac{\delta F}{\delta u}\frac{\delta G}{\delta v} - \frac{\delta F}{\delta v}\frac{\delta G}{\delta u} \ (\text{mod im}\, D). \tag{8}$$

We will also use the more customary representation $\int \left(\frac{\delta F}{\delta u}\frac{\delta G}{\delta v} - \frac{\delta F}{\delta v}\frac{\delta G}{\delta u}\right) dq$ of the bracket. In this picture the central objects are *Hamiltonians* $\int F$, $\int G$...

Remark: Everything we do here could be done for for general elements u and v in a commutative algebra, not necessarily $C^\infty(\mathbb{R})$, with a derivation.

We shall consider sequences of functionals $I_0, I_1, I_2, ...$ given by a (recursion or) *creation operator* C:

$$\text{(i)} \qquad I_n = CI_{n-1} = C^n I_0, \quad n \geq 0,$$

or, infinitesimally, $\delta I_n = C\delta I_{n-1}$. Throughout this paper, we will have

$$I_0 = uv. \tag{9}$$

The operator D given by

$$\mathsf{D}F := D^{-1}\left(v\frac{\delta F}{\delta v} - u\frac{\delta F}{\delta u}\right) = \{-qI_0, F\}, \tag{10}$$

is well defined on the space $\overline{\mathcal{A}}$ of equivalence classes of functionals that commute with I_0. We want the I_n to satisfy

$$\text{(ii)} \quad \mathsf{D}I_n = nI_{n-1}, \quad n \geq 0.$$

Together, properties (i) and (ii) yield a representation of the *Heisenberg algebra*: we have $[\mathsf{D}, C] = 1$ (the identity) on $\bigoplus_{n\geq 0} \mathbb{C} I_n$. D is the *annihilation operator*. There are traces of (ii) in Dickey's book [5], in connection with the KdV equation.

We also want the I_n to be *involutive*, i.e. to commute:

$$\text{(iii)} \qquad \{I_n, I_m\} = 0, \quad \text{all} \quad n, m \geq 0.$$

Properties (ii) and (iii) imply that the expected value of position, q, taken in the (ground) state I_0,

$$\langle q \rangle = \int q I_0 \, dq, \tag{11}$$

fulfills the *free Newton equations*

$$\frac{d^2 \langle q \rangle}{dt_n dt_m} = 0, \quad \text{all} \quad n, m \geq 0. \tag{12}$$

Here t_n is the 'time' obtained using the Hamiltonian I_n. Property (iii) means that for any m, n, $dI_n/dt_m = 0$ in the space of equivalent functionals: each I_n is a conservation law w.r.t. any choice of time t_m.

Define an auxiliary creation operator $\hat{C}$ by

$$\hat{C}\delta F := \left(\frac{\delta \hat{C} F}{\delta u}, \frac{\delta \hat{C} F}{\delta v} \right)^{\mathsf{T}} = \begin{pmatrix} -D & 0 \\ 0 & D \end{pmatrix} \delta F - 2\lambda \begin{pmatrix} v \\ u \end{pmatrix} \mathsf{D} F, \tag{13}$$

where λ is a (real or complex) parameter. (This is a slight adaption of [8].)

Let the sequence of functionals $\hat{I}_n$ be given by

$$\hat{I}_n = \hat{C}^n I_0, \quad n \geq 0. \tag{14}$$

Consider two special cases:

$\lambda = 0$ leads to the *free case* (we write $D^\dagger = -D$)

$$\hat{I}_n(0) = \tfrac{1}{2}(u_n v + (-1)^n u v_n) = \tfrac{1}{2}(D^n u \cdot v + u D^{\dagger n} v). \tag{15}$$

For real non-zero λ, say $\lambda = 1$, we get a version of the NLS (non-linear Schrödinger) hierarchy (Faddeev-Takhtajan [8]). $\hat{I}_0 = I_0$ and $\hat{I}_1$ are the same, whereas the next few are

$$\begin{aligned}
&\hat{I}_2 = \hat{I}_2(0) - u^2v^2, \quad \hat{I}_3 = \hat{I}_3(0) - \tfrac{3}{2}uv(u_1v - uv_1),\\
&\hat{I}_4 = \hat{I}_4(0) - uv(u_2v + uv_2) + 4uu_1vv_1 + 2u^3v^3,\\
&\hat{I}_5 = \hat{I}_5(0) + 5uv(u_2v_1 - u_1v_2) + 5u^2v^2(u_1v - uv_1),\\
&\hat{I}_6 = \hat{I}_6(0) - 3(uu_2v_1^2 + u_1^2vv_2) - 12uu_2vv_2 + 5u_1^2v_1^2\\
&\qquad - (u_2^2v^2 + u^2v_2^2) - 50u^2u_1v^2v_1 - 10uv(u_1^2v^2 + u^2v_1^2) - 5u^4v^4.
\end{aligned}$$

Here, $\hat{I}_2$ is the Hamiltonian for the NLS equations. I_3 leads to KdV upon putting $v \equiv 1$. The entire KdV hierarchy can be deduced from the odd-indexed $\hat{I}_n$.

We introduce an extended family $I_n = I_n(\lambda, \phi)$, $n \geq 0$ as follows: Let $\phi \in C^\infty(\mathbb{R})$ and put

$$\Lambda = D + \phi, \qquad \Lambda^\dagger = -D + \phi. \tag{16}$$

We note in passing that $[\Lambda, \Lambda^\dagger] = 2\phi'$ (as a multiplication operator). The case when $\phi = q$ (or a first order polynomial in q) gives the Heisenberg algebra.

We define

$$C\delta F = \begin{pmatrix} \Lambda^\dagger & 0 \\ 0 & \Lambda \end{pmatrix} \delta F - 2\lambda \begin{pmatrix} v \\ u \end{pmatrix} \mathsf{D}F \tag{17}$$

with the above requirement on F. One finds

$$I_1 = \hat{I}_1 + \phi\hat{I}_0 \qquad \text{and} \qquad I_2 = \hat{I}_2 + 2\phi\hat{I}_1 + \phi^2\hat{I}_0. \tag{18}$$

Below we shall prove

Theorem 1 *For $n \geq 3$, there are polynomials $\psi_n = \psi_n(\phi, \phi', ..., \phi^{(n-1)})$ of degree $n-2$, such that*

$$I_n = \sum_{k=0}^{n} \binom{n}{k} \left(\phi^k - \psi_k\right) \hat{I}_{n-k}, \quad n = 0,\, 1,\, 2, ... \tag{19}$$

By definition $\psi_0 = \psi_1 = \psi_2 = 0$.

The properties (ii) and (iii) hold in the general case:

Theorem 2 $\{I_n, I_m\} = 0$ *for all $n, m \geq 0$, and* $\mathsf{D}I_n = nI_{n-1}$ *for all $n \geq 0$.*

We shall also make use of the following result. Needless to say, it holds in the sense of equivalence of functionals.

Theorem 3 *For any $f \in C^\infty(\mathbb{C}^{n+1})$ we have*

$$\mathsf{D}\left(f(I_0, I_1, ..., I_n)\right) = \sum_{\nu=0}^{n} \frac{\partial f}{\partial I_\nu}(I_0, I_1, ..., I_n)\nu I_{\nu-1}. \tag{20}$$

This leads to a bundle where the 'quantum space' $\mathbb{C}[I_0, I_1, I_2, ...]$ of all polynomials in the variables (conservation laws) I_n, projects down to the 'classical space' $\mathbb{C}[I_1/I_0]$. The fibre is the kernel of D and may be identified with classical semi-invariants. We refer to the precise details in §5 below.

2 Preliminaries and background.

2.1 Symmetries and conservation laws for linear heat equations.

Assume that u and v satisfy

$$\dot{u} + \frac{1}{2}u'' - Vu = 0; \qquad -\dot{v} + \frac{1}{2}v'' - Vv = 0, \tag{21}$$

where V is a given, smooth, potential. Then

$$\frac{d}{dt}\int uv\, dq = 0, \tag{22}$$

assuming sufficently rapid decrease at infinity of u and v. Let $f = f(t, q)$ and put $Kf = \dot{f} + \frac{1}{2}f'' - Vf$. It is easy to show (see Brandão and Kolsrud [4]) that

$$\frac{d}{dt}\int f\, uv\, dq = \int \partial f\, uv\, dq = \int \partial^* f\, uv\, dq, \tag{23}$$

where

$$\partial f := \frac{1}{u}K(fu) = \dot{f} + \tfrac{1}{2}f'' + \frac{u'}{u}f, \tag{24}$$

$$\partial^* f := -\frac{1}{v}K^\dagger(fv) = \dot{f} - \tfrac{1}{2}f'' - \frac{v'}{v}f. \tag{25}$$

The first identity can be written

$$\partial = u^{-1} K u \quad \text{or} \quad u \partial u^{-1} = K. \tag{26}$$

Suppose the linear PDO $\Lambda = T\partial_t + Q\partial_q + U$ belongs to the *heat Lie algebra* of K:

$$[K, \Lambda] = K\Lambda - \Lambda K = \Phi \cdot K \tag{27}$$

for some function $\Phi = \Phi_\Lambda$. Using $u\partial u^{-1} = K$ we obtain

$$u\partial(\Lambda u/u) = K\Lambda u = ([K, \Lambda] + \Lambda K)u = (\Phi_\Lambda + \Lambda)Ku = 0. \tag{28}$$

This is an alternative way of expressing that $\Lambda u \cdot v$ is the density of a conservation law:

$$\frac{d}{dt}\int \Lambda u \cdot v \, dq = \int \partial(u^{-1}\Lambda u) \ \cdot uv \, dq = \int (\Phi_\Lambda + \Lambda)Ku \cdot uv \, dq = 0. \tag{29}$$

In more detail, the equation $\partial(\Lambda u/u) = 0$ above may be written

$$\partial\left(T\frac{\dot{u}}{u} + Q\frac{u'}{u} + U\right) = 0, \tag{30}$$

very much as in the classical case, where the Noether theorem leads to a constant of motion of the form $ET + Tp + U$, where $E = -\frac{1}{2}p^2 + V$ (Euclidean convention) and $p = \dot{q}$. The coefficients $\dot{u}u^{-1} = -\frac{1}{2}u''u^{-1} + V$ and $u'u^{-1}$ are, respectively, the *energy density* and the *momentum density* in a form that emphasises the backward motion, and the classical total time derivative is replaced by ∂.

The basic density is the ground state $I_0 = uv$, and for instance $u'v = \frac{u'}{u} \cdot I_0$ is an equivalent form for $\frac{1}{2}(u_1 v - u v_1) = \hat{I}_1(0)$. We write $\bar{p}$ for the moment density u'/u. Then we have the Euclidean Newton/Hamilton equations (cf Landau-Lifshitz [20], Eq. (19.3))

$$\partial q = \bar{p} = -H_{\bar{p}}, \qquad \partial p = V' = H_q, \tag{31}$$

with H denoting the Euclidean Hamiltonian $-\frac{1}{2}p^2 + V$. Similarly the energy density $\bar{E}$ satisfies $\partial \bar{E} = \dot{V}$.

Repeating the argument above one finds that for Λ_j in the Lie algebra of K and $s_j \geq 0$ integers, we have

$$\frac{d}{dt}\int \Lambda_1^{s_1} \cdots \Lambda_m^{s_m} u \cdot v \, dq = 0, \tag{32}$$

i.e. $\Lambda_1^{s_1} \cdots \Lambda_m^{s_m} u \cdot v$ is the density of a conservation law. Its probabilistic version is that $\Lambda_1^{s_1} \cdots \Lambda_m^{s_m} u/u$ is a martingale for a certain diffusion process.

2.2 The operator D.

Consider a slightly more general situation, in which our space is built from one independent variable q and m dependent variables u^α, $\alpha = 1, ..., m$. We define $\mathcal{A}$ as all smooth (C^∞) functions of the variables q and u^α_j, where $j = 0, ..., n$, so that the order n is arbitrary but finite.

Given a (canonical) vector field

$$X = a^\alpha \frac{\partial}{\partial u^\alpha}, \quad a^\alpha \in \mathcal{A} \tag{33}$$

together with its (infinite order) extension (prolongation), Ibragimov [10-12] or Olver [26],

$$\widetilde{X} = \sum_{k\geq 0} a^\alpha_k \frac{\partial}{\partial u^\alpha_k}, \quad a^\alpha_k = D^k a^\alpha. \tag{34}$$

(Any vector field as above lifts in a canonical way to a vector field on the appropriate jet bundle.) Summation over the repeated index $\alpha = 1, ..., m$ is understood.

Write Ξ for the variational counterpart of X:

$$\Xi = a^\alpha \frac{\delta}{\delta u^\alpha}. \tag{35}$$

Lemma 1 *For $I \in \mathcal{A}$ we have*

$$(\widetilde{X} - \Xi)I = D \sum_{k\geq 0} \hat{a}^\alpha_k \frac{\partial I}{\partial u^\alpha_{k+1}},$$

where $\hat{a}^\alpha_k$ denotes the differential operator

$$\hat{a}^\alpha_k := D^k a^\alpha - D^{k-1} a^\alpha D + ... + a^\alpha (-D)^k.$$

Proof: Fix I. It suffices to consider the case when I only depends on one variable u and its derivatives. Write $\partial_j I = \partial I / \partial u_j$. Then

$$\begin{aligned}(\widetilde{X} - \Xi)I = &\sum_{k=0}^{\infty} (D^k a \partial_k I - a(-D)^k \partial_k I) = (Da\partial_1 I + aD\partial_1 I)\\
+&(D^2 a\partial_2 I - aD^2\partial_2 I) + ... + (D^n a\partial_n I + (-1)^{n-1} D^n \partial_n I) + ...\\
=&D(a\partial_1 I + (Da\partial_2 I - aD\partial_2 I) + (D^2 a\partial_3 I - DaD\partial_3 I + aD^2\partial_3 I) + ...\end{aligned}$$

as stated.

We define the operator

$$\mathsf{D}_X := D^{-1}(\Xi - \widetilde{X}). \tag{36}$$

The following is a key result:

Theorem 4 *On the space of equivalence classes of functionals $\overline{\mathcal{A}} = \mathcal{A}/D\mathcal{A}$, we have*

$$\mathsf{D}_X = -\sum_{k\geq 0}(k+1)a_k^\alpha \frac{\partial}{\partial u_{k+1}^\alpha}. \tag{37}$$

In particular, D_X is a derivation on $\overline{\mathcal{A}}$.

Proof: Clearly

$$(D^k a^\alpha - D^{k-1}a^\alpha D + ... + a^\alpha(-D)^k)\frac{\partial I}{\partial u_{k+1}^\alpha}$$

is equivalent to

$$(k+1)D^k a^\alpha \frac{\partial I}{\partial u_{k+1}^\alpha}$$

modulo the image of D.

Corollary 1 *If $\widetilde{X}I = 0$, then*

$$\mathsf{D}_X I = D^{-1}\Xi I = D^{-1}a^\alpha \frac{\delta I}{\delta u^\alpha}. \tag{38}$$

In particular, for $X = v\partial/\partial v - u\partial/\partial u$ and $I_0 = uv$, we have

$$\mathsf{D}_X I = D^{-1}\left(v\frac{\delta I}{\delta v} - u\frac{\delta I}{\delta u}\right) = \mathsf{D}I = \{-qI_0, I\}, \tag{39}$$

provided $\widetilde{X}I = 0$.

Definition: Let $m = 2$. A differential function on the form constant times

$$u_0^{s_0}u_1^{s_1}\cdots u_N^{s_N}v_0^{t_0}v_1^{t_1}\cdots t_N^{t_N},$$

where $s_j, t_k \in \mathbb{N}$, is a *balanced monomial* if the number of u_js and v_ks are equal, i.e., if $\sum s_j = \sum t_k$. A finite sum of balanced monomials is a *balanced polynomial.*

Corollary 2 *Let $X = v\partial/\partial v - u\partial/\partial u$. Any functional of the form $I = f(A_1, ..., A_N)$, where f is C^∞ and A_j are balanced polynomials, satisfies $\widetilde{X}I = 0$. Consequently,*

$$\mathsf{D}I = \sum_{k\geq 0}(k+1)\left(u_{k+1}\frac{\partial I}{\partial u_k} - v_{k+1}\frac{\partial I}{\partial v_k}\right) \tag{40}$$

for such I.

Proof. It suffices to show that $\widetilde{X}F = 0$ for any balanced monomial F. Thus, consider $F = u^{s_0}u_1^{s_1}\cdots u_N^{s_N}v^{t_0}v_1^{t_1}\cdots v_N^{t_N}$ with $\sum s_n = \sum t_n$. Then

$$\widetilde{X}F := \sum_n \left(v_n\frac{\partial F}{\partial v_n} - u_n\frac{\partial F}{\partial u_n}\right) = \sum(t_n - s_n)F = 0.$$

Remark: All elements in the NLS hierarchy are balanced monomials.

3 Theorem 1.

3.1 Proof of Theorem 1.

We start with the case $\lambda = 0$, and consider a bit more generally, a sequence of functionals

$$T_n = \sum_{k=0}^{n}\binom{n}{k}\alpha_k\hat{I}_{n-k}(0), \quad n = 0, 1, 2, ...,$$

where $\alpha_0 \equiv 1, \alpha_1, \alpha_2, ...$ are given, smooth, functions, and, as above, $\hat{I}_s(0) = \frac{1}{2}(u_s v + (-1)^s u v_s)$, $s = 0, 1, 2, 3, ...$

We get

$$\frac{\delta T_n}{\delta v} = u_n + \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\left(\alpha_k u_{n-k} + D^{n-k}(\alpha_k u)\right),$$

which, upon resumming, becomes

$$
\begin{aligned}
&= u_n + n\alpha_1 u_{n-1} + \frac{1}{2}\sum_{k=2}^{n}\left(2\binom{n}{k}\alpha_k + \sum_{j=1}^{k-1}\binom{n}{k-j}\binom{n-k+j}{j}D^j\alpha_{k-j}\right)\\
&= u_n + n\alpha_1 u_{n-1} + \frac{1}{2}\sum_{k=2}^{n}\binom{n}{k}\left(2\alpha_k + \sum_{j=1}^{k-1}\binom{k}{j}D^j\alpha_{k-j}\right)u_{n-k}\\
&= u_n + n\alpha_1 u_{n-1} + \sum_{k=2}^{n}\binom{n}{k}A_k u_{n-k}.
\end{aligned}
$$

Choosing

$$\alpha_k =: \phi^k := \phi^k - \psi_k,$$

T_n becomes $I_n(0)$. Assume that $\psi_3, ..., \psi_{n-1}$ have been chosen so that

$$\frac{\delta I_n(0)}{\delta v} = u_n + \sum_{k=1}^{n-1}\Lambda^{k-1}\phi\cdot u_{n-k} + A_n u,$$

where

$$A_n = \phi^n - \psi_n + \frac{1}{2}\sum_{j=1}^{n-1}\binom{n}{j}D^j(\phi^{n-j} - \psi_{n-j}) = A_n^0 - \psi_n.$$

Our task is to choose ψ_n, satisfying the criteria in Theorem 1, such that $\phi^n - \psi_n = \Lambda^{n-1}\phi$. Before doing this, we remark that the coefficient

$$\frac{1}{2}\binom{n}{k}\left(2\alpha_k + \sum_{j=1}^{k-1}\binom{k}{j}D^j\alpha_{k-j}\right)$$

of u_{n-k} in the formula above only depends on n through the binomial coefficient $\binom{n}{k}$. It shows that $:\phi^k:$, the renormalisation of the k^{th} power of ϕ, is independent of n.

We note that both A_n^0 and $B_n \equiv \Lambda^{n-1}\phi$ are polynomials satisfying the criteria in Theorem 1. The choice $\psi_n = A_n^0 - B_n$ determines ψ_n uniquely.

In the general case, $I_n = \hat{I}_n(0) + J_{n-2}$, where J_{n-2} only depends on $u_j, v_j, :\phi^j:$ for $j \leq n-2$. The renormalisation term $:\phi^n:$ only occurs in the first term $\hat{I}_n(0)$, which we have already discussed. A similar argument works for $\delta I_n/\delta u$. Theorem 1 follows.

3.2 Comments on renormalisation.

We consider the linear case $\lambda = 0$. Variation of the space-time action $\iint L\,dtdq$, with (space-time) Lagrangian density

$$L = \frac{1}{2}(u\dot{v} - \dot{u}v) + \frac{1}{2}I_2, \tag{41}$$

leads to the Euler-Lagrange equations $\dot{u} = -\frac{1}{2}\frac{\delta I_2}{\delta v}$, $\quad \dot{v} = \frac{1}{2}\frac{\delta I_2}{\delta u}$. (See Ibragimov and Kolsrud [13].) In more detail:

$$\dot{u} + \frac{1}{2}u_2 + \phi u_1 + \frac{1}{2}(\phi^2 + \phi')u = \dot{u} - Hu = 0; \tag{42}$$

$$-\dot{v} + \frac{1}{2}v_2 - \phi v_1 + \frac{1}{2}(\phi^2 - \phi')v = -\dot{v} - H^\dagger v = 0. \tag{43}$$

Remark: The variational principle appears for the Schrödinger equation in Goldstein [8], and the same trick is used in classical mechanics in Morse and Feshbach [25]. It is related to the Hilbert integral, in, e.g. [1].

The Hamiltonian H is evidently unsymmetric, and $H \to H^\dagger$ precisely when $\phi \to -\phi$. In the non-linear case $\lambda \neq 0$ the equations become

$$\dot{u} - Hu = \lambda u^2 v; \qquad -\dot{v} - H^\dagger v = \lambda u v^2. \tag{44}$$

Writing $:\phi^n: \equiv \ \phi^n - \psi_n$, as above, one finds

$$:\phi^3: = \phi^3 - \tfrac{1}{2}\phi'', \quad :\phi^4:= \phi^4 - 2\phi\phi'' - 3\phi'^2, \tag{45}$$

$$:\phi^5: = \phi^5 - 5\phi^2\phi'' - 15\phi\phi'^2, \tag{46}$$

$$:\phi^6: = \phi^6 - 40\phi^3\phi'' - 45\phi^2\phi'^2 + 6\phi\phi^{\text{iv}} + 30\phi'\phi''' + 10\phi''^2, \ldots. \tag{47}$$

Examples: It is known that the Lie algebra is maximal when the potential is quadratic in q. Then the Lie algebra contains an sl_2, in addition to the Heisenberg algebra. The latter occurs only in this case, and the case of inverse square potential, $V = c/q^2$.

(i) In the case $\phi = cq$ no renormalisation occurs for $n = 3$. For $c = 1$ we get

$$:q^n:= q^n - 3\binom{n}{4}q^{n-4}, \qquad n \geq 4. \tag{48}$$

This is related to the harmonic oscillator and the $: q^n :$ share some properties with Hermite polynomials, notably $D : q^n := n \ : q^{n-1} :$. However, to get a potential with correct sign we must take c imaginary. (For *Gaussian* renormalisation, see Simon [29].)

(ii) The choice $\phi = c/q$ is related to the inverse square potential. For $c = 1$ we have

$$:q^{-n}: = \lambda_n q^{-n}, \quad n \geq 0. \tag{49}$$

The first few odd-indexed $:q^{-n}:$ vanish. In particular, $:q^{-3}:= 0$. This happens only in this case. Here, one can keep c real and get correct sign for one, but not both, of the potentials, $\frac{1}{2}(\phi^2 \pm \phi')$. Many formulae become particularly simple in this case, because $\Lambda(1/q) = 0$.

Remark: Incidentally, $u = c_1/q$, $v = c_2/q$ solves the stationary NLS equation $u'' - 2u^2v = 0$, $v'' - 2uv^2 = 0$, provided $c_1c_2 = 1$. The same holds for the stationary KdV system obtained from $\hat{I}_3$, viz. $u''' - 6uu'v = 0$, $v''' - 6uvv' = 0$, and possibly for corresponding higher order equations. Cf. Treves [32].

4 Proof of Theorem 2.

4.1 Involutivity.

We first show that C is symmetric w.r.t. the bracket.

Lemma 2 *Assume two functionals F and G commute with I_0. Then*

$$\{CF, G\} = -\{CG, F\} = \{F, CG\}. \tag{50}$$

Proof. The second identity is a consequence of the bracket being antisymmetric. To prove the first identity, we write $\{CF, G\} =$

$$\begin{aligned}
&\int \left(\left(\Lambda^\dagger \frac{\delta F}{\delta u} - 2\lambda v \mathsf{D} F \right) \frac{\delta G}{\delta v} - \left(\Lambda \frac{\delta F}{\delta v} - 2\lambda u \mathsf{D} F \right) \frac{\delta G}{\delta u} \right) dq \\
= &\int \left(\left(\frac{\delta F}{\delta u} \Lambda \frac{\delta G}{\delta v} - \frac{\delta F}{\delta v} \Lambda^\dagger \frac{\delta G}{\delta u} \right) - 2\lambda \mathsf{D} F \left(v \frac{\delta G}{\delta v} - u \frac{\delta G}{\delta u} \right) \right) dq \\
= &\int \left(\frac{\delta F}{\delta u} \Lambda \frac{\delta G}{\delta v} - \frac{\delta F}{\delta v} \Lambda^\dagger \frac{\delta G}{\delta u} - 2\lambda \mathsf{D} F D \mathsf{D} G \right) dq,
\end{aligned}$$

where we have used the definition $\mathsf{D} = D^{-1}(v\delta/\delta v - u\delta/\delta u)$. Performing a partial integration in the last term, we clearly get $-\{CG, F\}$. This proves our claim.

To prove that all I_n commute, we assume that

$$\{I_j, I_k\} = 0, \quad 0 \le j, k \le n, \tag{51}$$

and prove that it can be extended to the first $n+1$ conservation laws. If $j < n$, the above observation shows

$$\{I_{n+1}, I_j\} = \{CI_n, I_j\} = \{I_n, I_{j+1}\},$$

which vanishes by hypothesis. The identity also shows that $\{I_{n+1}, I_n\} = \{I_n, I_{n+1}\}$, which since the bracket is anti-symmetric allows us to conclude that, indeed also $\{I_{n+1}, I_n\} = 0$.

4.2 Proof of $\mathsf{D}I_n = nI_{n-1}$.

It is clear from the results in §2.2 that for any balanced I we have $\mathsf{D}(f(q)I) = f(q)\mathsf{D}I$. Hence the formula in Theorem 1 leads to

$$\mathsf{D}I_n = \sum_{k=0}^{n} \binom{n}{k} :\phi^k: \mathsf{D}\widehat{I}_{n-k} = \sum_{k=0}^{n} \binom{n}{k} :\phi^k: (n-k)\widehat{I}_{n-k-1} = nI_{n-1},$$

provided the $\widehat{I}_n$ from the NLS-hierarchy fulfill (ii). This is what we shall prove. We drop the hats from now on, and assume that $\lambda = 1$.

Write $a_n := \mathsf{D}I_n$. In general,

$$a_n = D^{-1}\left(v\frac{\delta I_n}{\delta v} - u\frac{\delta I_n}{\delta u}\right) = \int -q\left(v\frac{\delta I_n}{\delta v} - u\frac{\delta I_n}{\delta u}\right) dq = \int -q a_n' \, dq.$$

Use of the creation operator C and partial integration leads to the relation

$$\begin{aligned} a_{n+1}' =& u^{(n+1)}v + (-1)^n uv^{(n+1)} \\ & -2\Big((a_1u)^{(n-1)}v + (-1)^n u(a_1v)^{(n-1)} + \dots \\ & \quad + (a_{n-2}u)''v - u(a_{n-2}v)'' + (a_{n-1}u)'v + u(a_{n-1}v)'\Big) \\ & = A_{n+1} + A_{n-1} + \dots\dots \end{aligned}$$

where the index on the right refers to the total number of derivatives.

Assume now that $a_k = kI_{k-1}$ for all $k \leq n$. The terms of lowest order will come from

$$-2\big((a_{n-1}u)'v + u(a_{n-1}v)'\big)$$

if n is even, and from

$$-2\big((a_{n-2}u)''v - u(a_{n-2}v)'' + (a_{n-1}u)'v + u(a_{n-1}v)'\big)$$

if n is odd.

In the former case, the hypothesis yields

$$a'_{n+1} = -2(n-1)((I_{n-2}uv)' + I'_{n-2}uv) + \text{ higher order terms.}$$

In general,

$$I_{2m} = c_{2m}(uv)^{m+1} + \text{ higher order terms,}$$

where the coefficient is (if $(-1)!! = 1$)

$$c_{2m} = (-2)^m \frac{(2m-1)!!}{(m+1)!}, \qquad m = 0, 1, 2, \dots$$

This expression can be found using the following formulae for C^2:

$$\begin{aligned}
\frac{\delta I_{k+2}}{\delta u} &= D^2 \frac{\delta I_k}{\delta u} + 2(a_k v)' - 2a_{k+1}v, \\
\frac{\delta I_{k+2}}{\delta v} &= D^2 \frac{\delta I_k}{\delta v} - 2(a_k u)' - 2a_{k+1}u.
\end{aligned}$$

With $n = 2m$, and writing $s := uv$, the terms of lowest order are

$$\begin{aligned}
&-2(2m-1)c_{2(m-1)}\big((s^{m+1})' + (s^m)'s\big) \\
=&-2(2m-1)c_{2(m-1)}(2m+1)s^m s' \\
=&(2m+1)(-2)^m \frac{(2m-1)!!}{m!} \frac{(s^{m+1})'}{m+1} = (2m+1)(c_{2m}s^{m+1})',
\end{aligned}$$

which proves the assertion in this case.

In the case when n is *odd*, $n = 2m+1$, the lowest order terms for a'_{2m+2} are obtained from

$$\begin{aligned}
-2\big(&2m(I_{2m-1}s)' \\
&+2mI'_{2m-1}s + (2m-1)I_{2m-2}a \\
&+(2m-1)(I_{2m-2}a)'\big),
\end{aligned}$$

where, in addition to $s = uv$, we have written $a := u'v - uv'$. In general,

$$I_{2m+1} = c_{2m+1} s^m a + \text{ h. o. t.}$$

for some constant c_{2m+1}. Hence the middle terms above are

$$\begin{aligned} & 2mc_{2m-1}(s^{m-1}a)'s + (2m-1)c_{2m-2}(s^m)'a \\ = \ & (2mc_{2m-1}(m-1) + (2m-1)mc_{2m-2})s^{m-1}s'a + 2mc_{2m-1}s^m a' \\ = \ & (2(m-1)c_{2m-1} + (2m-1)c_{2m-2})(s^m)'a + 2mc_{2m-1}s^m a' \\ = \ & 2mc_{2m-1}(s^m a)', \end{aligned}$$

provided $2(m-1)c_{2m-1} + (2m-1)c_{2m-2} = 2mc_{2m-1}$, i.e.

$$c_{2m-1} = \frac{2m-1}{2} c_{2(m-1)}.$$

One may deduce this formula from the formula for c_{2m} together with the formulae for C^2 displayed above.

The lowest order terms become

$$-2(2 \cdot 2mc_{2m-1} + (2m-1)c_{2m-2})(s^m a)'.$$

The coefficient can be written

$$\begin{aligned} & -2(2m-1)(2m+1)c_{2m-2} \\ & = 2(m+1) \cdot \frac{2m+1}{2}(-2)^m \frac{(2m-1)!!}{(m+1)!} = 2(m+1)c_{2m+1}, \end{aligned}$$

which proves our claim

$$a'_{2(m+1)} = 2(m+1)c_{2m+1}(s^m a)' + \text{ h. o. t.}$$

By induction, we may assume that all terms of order strictly less than the highest order, viz. $n+1$, satisfy the corresponding identity. It remains to prove that $J_n := \frac{1}{2}(u^{(n)}v + (-1)^n uv^{(n)})$ fulfil

$$\mathsf{D}J_n = nJ_{n-1} \qquad \text{for all} \quad n.$$

This is the relation $\mathsf{D}I_n = nI_{n-1}$ in the free case. It follows immediately from noting that

$$u^{(n)}v - (-1)^n uv^{(n)} = D\left(u^{(n-1)}v - u^{(n-2)}v' + \ldots.. + (-1)^{n-1}uv^{(n-1)}\right),$$

with each term within the parentheses being equivalent to I_{n-1}.

5 $\mathbb{C}[I_0, I_1, ...]$ and D-invariant polynomials.

5.1 General setting.

Suppose we are given

$$I_0, I_1, I_2, ... \tag{52}$$

and a derivation D for which

$$\mathsf{D}I_n = nI_{n-1}, \quad n \geq 0. \tag{53}$$

Replacing I_n by $\overline{I}_n := I_n/I_0$, one finds that once again, $\mathsf{D}\overline{I}_n = n\overline{I}_{n-1}$, and $\overline{I}_0 = 1$. We may therefore assume $I_0 \equiv 1$.

Each I_n is assigned the *degree* n. We may then define the degree of a *monomial*

$$M_\alpha = I_1^{\alpha_1} I_2^{\alpha_2} \cdots I_s^{\alpha_s}, \quad \alpha_i \in \mathbb{N}, \tag{54}$$

as

$$\deg M_\alpha := \sum_{j=1}^{s} j\alpha_j = ||\alpha||. \tag{55}$$

A linear combination of monomials of the same degree,

$$P = \sum_{||\alpha||=N} c_\alpha M_\alpha, \tag{56}$$

is a *homogeneous polynomial* of degree N. General polynomials in $I_1, I_2, ...$ are linear combinations of homogeneous polynomials. This way, the ring of polynomials $\mathsf{P} := \mathbb{C}[I_1, I_2, ...]$ (the field of scalars does not seem so important) gets a natural grading by the degree:

$$\mathsf{P} = \bigoplus_{N=0}^{\infty} \mathsf{P}_N, \tag{57}$$

where P_N denotes all homogeneous polynomials of degree N.

D being a derivation,, we have

$$\mathsf{D}\{f(I_1, I_2, ..., I_n)\} = \sum_{j=1}^{n} \frac{\partial f}{\partial I_j}(I_1, I_2, ..., I_n)\mathsf{D}I_j \tag{58}$$

for every polynomial f.

Define polynomials K_N, $N \geq 2$, by the formula

$$K_N = \sum_{\nu=0}^{N-2} (-1)^\nu \binom{N}{\nu} I_1^\nu I_{N-\nu} + (-1)^{N-1}(N-1) I_1^N. \tag{59}$$

Then D annihilates all K_N:

$$\mathsf{D} K_N = 0, \quad N \geq 2, \tag{60}$$

as one easily checks. Together, all the K_N form an *algebra of invariant polynomials*. (Since D is a derivation, it annihilates the whole algebra, according to the Leibniz rule.)

Theorem 5 P_N *has the decomposition*

$$\mathsf{P}_N = \mathsf{H}_N \oplus \mathsf{V}_N, \tag{61}$$

where

$$\mathsf{H}_N := I_1 \mathsf{P}_{N-1} \quad \textit{and} \quad \mathsf{V}_N = \mathbb{C}[K_2, K_3, ..., K_N] \cap \mathsf{P}_N. \tag{62}$$

Proof: The monomials in P_N are divided into two classes according to whether the monomial contains a factor I_1 or not. Clearly, the first case corresponds to the space H_N, as defined above.

The remaining monomials can be identified with those partitions of N, for which all summands fulfil $2 \leq \lambda_i \leq N$. The trivial partition, i.e. N itself, is identified with K_N. The other partitions satisfy $2 \leq \lambda_i \leq N-2$, and can be identified with $K_{\lambda_1} K_{\lambda_2} \cdots K_{\lambda_s}$. (Typically there are repetitions of the K_j.) This proves the theorem.

The theorem leads to

$$\mathsf{P}_N / \mathsf{V}_N \simeq \mathsf{H}_N. \tag{63}$$

From $K_2 = 0$ follows $I_2 = I_1^2$ and $K_3 = 0$ leads to $I_3 = 3I_1 I_2 - 2I_1^3 = I_1^3$ etc. We get $I_n = I_1^n$ for $0 \leq n \leq N$. In particular, H_N can be identified with the one-dimensional space that I_1^N generates.

The result as $N \to \infty$ may be written

Corollary 3 $\mathbb{C}[I_1, I_2, ...]/\mathbb{C}[K_2, K_3, ...] = \mathbb{C}[I_1]$.

The invariant polynomials have integer coefficients and it holds that $\mathbb{Z}[I_1, I_2, ...]/\mathbb{Z}[K_2, K_3, ...] = \mathbb{Z}[I_1]$.

5.2 Invariants and semi-invariants.

To see how the K_N arise, we start from the quotient I_2/I_1^2, homogeneous of degree zero. Acting on it by the vector field $I_1\mathsf{D}$ we get a new function with the same homogeneity. The obtained relation may be written as

$$-I_1\mathsf{D}(I_2/I_1^2) = 2K_2/I_1^2. \tag{64}$$

More generally, one finds

$$-I_1\mathsf{D}\left(\frac{I_{n+1}}{I_1^{n+1}}\right) = (n+1)\sum_{j=0}^{n-1}\binom{n}{j}\frac{K_{n+1-j}}{I_1^{n+1-j}}. \tag{65}$$

This can be used to obtain a kind of generating function relation between the I_n and the K_N.

If we bring back I_0, the first few K_N are

$$K_2 = I_0I_2 - I_1^2, \quad K_3 = I_0^2I_3 - 3I_0I_1I_2 + 2I_1^3, \tag{66}$$
$$K_4 = I_0^3I_4 - 4I_0^2I_1I_3 + 6I_0I_1^2I_2 - 3I_1^4 \tag{67}$$

These expressions are *semi-invariants*, or *relative invariants*, related to *forms*, i.e. homogeneous polynomials in two variables, and projective representations of $GL(2,\mathbb{R})$ or $SL(2,\mathbb{R})$. We refer to Gurevich [9] , Ibragimov [10], and Olver [27, 28] for more about this classical, fascinating subject.

K_2, the *discriminant*, is a true invariant. It can be written $\begin{vmatrix} I_0 & I_1 \\ I_1 & I_2 \end{vmatrix}$.

Instead of K_4 we could have chosen the well-known invariant (for quartic polynomials) $\widehat{K}_4 := I_0I_4 - 4I_1I_3 + 3I_2^2$. Except for the (irrelevant) factor I_0^2, K_4 and $\widehat{K}_4$ differ by a multiple of K_2^2.

The 3×3 determinant with rows, from the top, $(I_0\ I_1\ I_2)$, $(I_1\ I_2\ I_3)$, $(I_2\ I_3\ I_4)$, is another well-known invariant related to quartic polynomials (Olver, [27], p. 97). It is a linear combination of K_2^3, K_3^2 and K_2K_4.

For N even, there is an alternative choice of K_N, , with $\widehat{K}_4$ as a special case, namely

$$\widehat{K}_{2n} := \sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}I_kI_{2n-k} + \frac{(-1)^n}{2}\binom{2n}{n}I_n^2.$$

5.3 Proof of Theorem 3.

Every I_n is a balanced polynomial with q-dependent coefficients. Hence the formula in Corollary 2 applies. It shows that D is a derivation on functionals of the form $f(I_0, ..., I_n)$. Thus

$$\mathsf{D}\{f(I_0, ..., I_n)\} = \sum_{\nu=0}^{n} \partial_\nu f(I_0, .., I_n)\mathsf{D}I_\nu,$$

which according to Theorem 2 yields the desired result.

Corollary 4 *Let $(I_n)_0^\infty$ be given by Theorem 1. Then*

$$\mathbb{C}[I_0, I_1, I_2, ...]/\mathbb{C}[K_2, K_3, ...] = \mathbb{C}[I_0, I_1, ...]/ker\,\mathsf{D} = \mathbb{C}[I_1/I_0] \simeq \mathbb{C}[p].$$

5.4 A final remark.

One may ask what happens if we switch between I_1 and $I_1^* := -qI_0$. The sequence $I_n^* := -q^n I_0$ forms, of course, an abelian algebra. Using $\mathsf{D}^* := ad_{I_1}$, we get the same derivation property as above. In this case, however, the recursion operator $I_n^* \to I_{n+1}^*$ is trivial.

Acknowledgements: This article was written in memory of my friend Augusto Brandão Correia, 1961-2004, co-author of several papers, and discussion partner for many years.

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