AN EXTENSION OF THE CLASSICAL CONTINUITY CORRECTION

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Abstract. We define a variable continuity correction and give an asymptotic approximation for it. The approximation is an extension of the classical continuity correction of 1/2; it takes values in the open unit interval (0,1). The result is based on an approximation of a sum of normal densities that is claimed to provide an asymptotic expansion, and on a modified expression for the Maclaurin expansion of a composition of two functions, of which one is the normal distribution function.

1. Introduction

Continuity correction is a method that reduces the errors that appear when a discrete distribution is approximated by a continuous one. The classical continuity correction of 1/2 is based on ad-hoc arguments. We replace these arguments by a strict definition, and derive an asymptotic approximation of the newly defined continuity correction.

We consider the situation when the distribution of a discrete random variable X is approximated by the aid of a normal distribution with mean μ and standard deviation σ. The corresponding continuous random variable is denoted ξ. The density of the normal distribution at n is then given by $\varphi(y(n))/\sigma$, where φ denotes the normal density function $\varphi(y) = \exp(-y^2/2)/\sqrt{2\pi}$, and where we write

$$y(n) = \frac{n-\mu}{\sigma}. \tag{1.1}$$

The approximation process can be interpreted as consisting of two separate stages. The first stage gives an approximation of the discrete probability that X takes the value n, while the second one approximates the distribution function $F_X(k) = P\{X \le k\}$ of X. If X takes integer values, then the first stage of the approximation process leads to the specific approximation

$$P\{X = n\} \approx \frac{\alpha}{\sigma}\varphi(y(n)), \tag{1.2}$$

where α is constant, on some set of n-values. In the second stage we need an approximation of the normal density $\varphi(y(n))/\sigma$ in terms of

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the normal distribution function $\Phi(y) = \int_{-\infty}^{y} \varphi(x)\,dx$. The classical continuity correction of 1/2 is based on the approximation

$$\frac{1}{\sigma}\varphi(y(n)) \approx \Phi(y(n+1/2)) - \Phi(y(n-1/2)), \tag{1.3}$$

see Maxwell (1982). It leads to the following approximation of the distribution function for X:

$$F_X(k) \approx \alpha\Phi(y(k+1/2)), \tag{1.4}$$

We show below that the ad-hoc approximation in (1.3) is improved by our approach.

We notice that the discreteness of the distribution is maintained in the first stage of the approximation process, and that the transition from discreteness to continuity takes place at the second stage.

The error introduced in the second stage of approximation can be obliterated by defining a continuity correction $C(k)$ that depends on the argument k, instead of being constant as in the classical case. To achieve this, we define the continuity correction $C(k)$ to be the solution of the following relation:

$$\frac{1}{\sigma}\sum_{n=-\infty}^{k} \varphi(y(n)) = \Phi(y(k+C(k)). \tag{1.5}$$

This definition of the continuity correction does not involve any approximation, contrary to the case with the classical continuity correction described above. However, approximations are needed to describe the solution of (1.5). We derive an approximation of $C(k)$, which is asymptotic (except possibly in the extreme tails of the distribution of ξ) as $\sigma \to \infty$. Our main result takes the form

$$C(k) = G(z(k)) + \mathrm{O}\left(\frac{1}{\sigma^2}\right), \quad z(k) = \mathrm{O}(1), \quad \sigma \to \infty, \tag{1.6}$$

where the functions G and z are defined as follows:

$$G(z) = \begin{cases} \dfrac{1}{z}\log\dfrac{\exp(z)-1}{z}, & z \neq 0, \\ \dfrac{1}{2}, & z = 0, \end{cases} \tag{1.7}$$

and

$$z(n) = \frac{n-\mu}{\sigma^2} = \frac{y(n)}{\sigma}. \tag{1.8}$$

The function G is defined and analytic for all real values of its argument. It is monotonically increasing, satisfies the relation $G(z) + G(-z) = 1$, and takes values in the open unit interval $(0, 1)$. Thus, the approximation $G(z(k))$ of the new continuity correction takes values in the same unit interval. A plot of the function G is given in Figure 1. It follows from (1.7) and (1.8) that our approximation of $C(k)$ coincides with the classical continuity correction for $k = \mu$, since $z(\mu) = 0$ and $G(0) = 1/2$.

It is useful to note that the two quantities $y(n)$ and $z(n)$ take different orders of magnitude in the body and the tails of the distribution of the normal random variable ξ. We say that n is in the body of the distribution of ξ if the distance between n and the mean μ is a finite number of standard deviations σ. Translating this to asymptotic concepts, we find that $y(n) = \mathrm{O}(1)$ as $\sigma \to \infty$. Since $z(n) = y(n)/\sigma$, we conclude that $z(n) = \mathrm{o}(1)$ in the body of the distribution of ξ when σ is large. In the tails we find on the contrary that $y(n) \to \pm\infty$ and that $z(n)$ is at least of the order of 1. The restriction in (1.6) that $z(k) = \mathrm{O}(1)$ excludes only the extreme tails where $z(k)$ is not bounded.

We note that $G(z(k))$ is close to the value $1/2$ when $z(k)$ is small, as it is in the body of the distribution of ξ. Indeed, we derive the following result:

(1.9)
$$C(k) = \frac{1}{2} + \frac{1}{24} z(k) + \mathrm{O}((z(k))^3) + \mathrm{O}\left(\frac{1}{\sigma^2}\right), \quad z(k) = \mathrm{o}(1), \quad \sigma \to \infty.$$

The approximation in (1.3) that leads to the classical continuity correction can be shown to be an asymptotic approximation, as $\sigma \to \infty$, in the body of the distribution of ξ, but not in its tails. In contrast we can use our newly defined continuity correction to derive the following approximation:

$$\frac{1}{\sigma}\varphi(y(n)) \approx \Phi(y(n + G(z(n)))) - \Phi(y(n - 1 + G(z(n-1)))). \tag{1.10}$$

This approximation is asymptotic for a larger range of n-values, namely both in the body of the distribution of ξ, and in those parts of the tails where $z(k) = \mathrm{O}(1)$.

The derivation of (1.6) is based on two results. The first one is an asymptotic expansion of the sum $\sum_{n=-\infty}^{k} \varphi(y(n))/\sigma$ as $\sigma \to \infty$. It can be expressed as follows:

$$\begin{aligned}\frac{1}{\sigma}\sum_{n=-\infty}^{k} \varphi(y(n)) &\sim \Phi(y(k)) \\ &+ \frac{1}{\sigma}\varphi(y(k))\left[1 - \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{1}{2\sigma^2}\right)^m G_1^{(2m)}(z(k))\right], \quad \sigma \to \infty,\end{aligned} \tag{1.11}$$

The second result is a modification of the Maclaurin series of $\Phi(y(k+x))$ in x. It takes the following form:

(1.12)
$$\Phi(y(k+x)) = \Phi(y(k)) + \frac{1}{\sigma}\varphi(y(k)) \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{1}{2\sigma^2}\right)^m \frac{\partial^{2m}}{\partial z^{2m}} G_2(x, z(k)).$$

Both of these results are believed to be new.

We define the two functions G_1 and G_2 that appear here. G_1 is defined by

$$G_1(z) = \begin{cases} \dfrac{1}{\exp(z)-1} - \dfrac{1}{z} + 1, & z \neq 0, \\ \dfrac{1}{2}, & z = 0. \end{cases} \tag{1.13}$$

This function is defined and analytic for all real values of its argument. It is monotonically increasing, takes values in the open interval $(0,1)$, and satisfies the relation $G_1(z) + G_1(-z) = 1$. A plot of G_1 is given in Figure 1.

Furthermore, the function G_2 is given by

$$G_2(x,z) = \begin{cases} \dfrac{1-\exp(-xz)}{z}, & z \neq 0, \\ x, & z = 0. \end{cases} \tag{1.14}$$

The function G_2 is defined and analytic for all real values of x and z. Clearly, it is identically equal to 0 if $x = 0$. If $x \neq 0$ then the function G_2 has the same sign as x, and is a monotonically decreasing function of z.

Note the similarities between the sums over m in the right-hand sides of (1.11) and (1.12)!

We give a derivation of the Maclaurin expansion of (1.12) in Section 2. The claim that the right-hand side of (1.11) provides an asymptotic expansion has not been established rigorously, and is therefore left as an open problem. A formal, but nonrigorous, derivation is given in Section 3. An early derivation of a related result of similar caliber has been given by Nåsell (1996). The continuity correction result given in (1.6) is derived in Section 4. Numerical support for the claim that (1.11) gives an asymptotic expansion is given in Section 5.

2. Derivation of the modified Maclaurin expansion in (1.12)

We turn now to a derivation of the expression in (1.12). Thus, we wish to show that

$$\Phi(y(k+x)) = \Phi(y(k)) + \frac{1}{\sigma}\varphi(y(k))g_2(x, z(k), \sigma), \tag{2.1}$$

where

$$g_2(x,z,\sigma) = \sum_{m=0}^{\infty} \frac{1}{m!}\left(-\frac{1}{2\sigma^2}\right)^m \frac{\partial^{2m}}{\partial z^{2m}} G_2(x,z). \tag{2.2}$$

Since the function $\Phi(y(k+x))$ is analytic is analytic in x, we can write its Maclaurin expaansion as

$$\Phi(y(k+x)) = \Phi(y(k)) + \sum_{n=1}^{\infty} \frac{x^n}{n!\sigma^n} \varphi^{(n-1)}(y(k)). \tag{2.3}$$

Here we make use of the fact that the derivative of order n of the normal density function $\varphi(y)$ with respect to y is equal to the density itself, multiplied by a polynomial of degree n. The polynomial that appears here is known as a Hermite polynomial $He_n(y)$ with weight function $\exp(-y^2/2)$. This polynomial should not be confused with the polynomial $H_n(y)$, which is defined with respect to the weight function $\exp(-y^2)$, and which also is referred to as a Hermite polynomial. The derivatives of $\varphi(y)$ can be expressed in terms of the Hermite polynomials $He_n(y)$ as follows:

$$\varphi^{(n)}(y) = (-1)^n He_n(y)\varphi(y). \tag{2.4}$$

The first few Hermite polynomials are $He_0(y) = 1$, $He_1(y) = y$, $He_2(y) = y^2 - 1$, $He_3(y) = y^3 - 3y$. The general expression for the Hermite polynomial of arbitrary order n can be written

$$He_n(y) = \sum_{m=0}^{[n/2]} \frac{(-1)^m}{m!2^m} \frac{n!}{(n-2m)!} y^{n-2m}, \tag{2.5}$$

see e.g. Abramowitz and Stegun (1965).

By introducing the Hermite polynomials we can write

$$\Phi(y(k+x)) = \Phi(y(k)) + \frac{1}{\sigma} g_1(x, z(k), \sigma)\varphi(y(k)), \tag{2.6}$$

where

$$g_1(x, z, \sigma) = -\sigma \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{x}{\sigma}\right)^n He_{n-1}(\sigma z). \tag{2.7}$$

It remains to show that $g_1(x, z, \sigma) = g_2(x, z, \sigma)$. To do this, we express both of these two functions as double sums. For the function g_1 we insert the explicit expression (2.5) for the Hermite polynomial $He_n(y)$ into the defining relation (2.7) to get

$$\begin{aligned} g_1(x, z, \sigma) = -\sum_{n=1}^{\infty} \frac{1}{n!}(-x)^n \\ \cdot \sum_{m=0}^{[(n-1)/2]} \frac{1}{m!} \left(-\frac{1}{2\sigma^2}\right)^m \frac{(n-1)!}{(n-1-2m)!} z^{n-1-2m}. \end{aligned} \tag{2.8}$$

To express g_2 as a double sum we note first that the function G_2 and its partial derivatives with respect to z can be written as follows:

$$G_2(x,z) = x\sum_{n=1}^{\infty}\frac{(-xz)^{n-1}}{n!}, \tag{2.9}$$

$$\frac{\partial^n}{\partial z^n}G_2(x,z) = (-1)^n x^{n+1}\sum_{k=n+1}^{\infty}\frac{(k-1)!}{(k-n-1)!}\frac{(-xz)^{k-n-1}}{k!}. \tag{2.10}$$

Inserting this expression for the partial derivative of G_2 with respect to z into the defining relation (2.2) for g_2 gives

$$\begin{aligned} g_2(x,z,\sigma) = -\sum_{m=0}^{\infty}\frac{1}{m!}\left(-\frac{1}{2\sigma^2}\right)^m \\ \sum_{n=2m+1}^{\infty}\frac{(n-1)!}{(n-1-2m)!}\frac{(-x)^n}{n!}z^{n-2m-1}. \end{aligned} \tag{2.11}$$

It is straightforward to change the order of summation of either of these expressions for g_1 and g_2 to show that they are equal. It remains to justify this interchange of order of summation. This is allowed in this case, since both of the double sums converge absolutely.

To show that the double sum appearing in the right-hand side of (2.8) converges absolutely we define

$$\varphi_p(y) = \frac{1}{\varphi(y)} = \sqrt{2\pi}\exp(y^2/2), \tag{2.12}$$

$$\Phi_p(y) = \int_0^y \varphi_p(t)\,dt, \tag{2.13}$$

$$Hep_n(y) = \sum_{m=0}^{[n/2]}\frac{1}{m!2^m}\frac{n!}{(n-2m)!}y^{n-2m}. \tag{2.14}$$

It is then readily shown that

$$\varphi_p^{(n)}(y) = Hep_n(y)\varphi_p(y). \tag{2.15}$$

Since $\Phi(y(\cdot))$ is analytic, we find that

$$\begin{aligned} \Phi_p(y(i+x)) &= \Phi_p(y(i)) + \sum_{n=1}^{\infty}\frac{x^n}{n!\sigma^n}\varphi_p^{(n-1)}(y(i)) \\ &= \Phi_p(y(i)) + \frac{1}{\sigma}g_3(x,z(i),\sigma)\varphi_p(y(i)), \end{aligned} \tag{2.16}$$

where

$$g_3(x,z,\sigma) = \sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{m=0}^{[(n-1)/n]}\frac{1}{m!}\left(\frac{1}{2\sigma^2}\right)^m\frac{(n-1)!}{(n-1-2m)!}z^{n-1-2m}. \tag{2.17}$$

We note now that $g_3(|x|, |z|, |\sigma|)$ equals the sum of the absolute values of the individual terms that make up g_1. Since the former sum converges, we conclude that the double sum in the right-hand side of (2.8) converges absolutely.

3. Formal derivation of the asymptotic expansion in (1.11)

We claim that the approximation given in (1.11) is indeed an asymptotic expansion. This claim has, however, not been established rigorously. The proof of this claim is left as an open problem. Strong numerical support for this claim is given below in Section 5. The present section is used to indicate a formal derivation of the result in (1.11). We emphasize that all steps in this derivation are not mathematically justified.

We start out by quoting the Euler-Maclaurin formula, which can be used to approximate a sum of functions. Following Olver (1974), it can be written as follows

$$\sum_{n=j}^{k} f(n) = \int_j^k f(x)\,dx + \frac{1}{2}[f(j) + f(k)] + S_f(k, M) - S_f(j, M) + R_M, \tag{3.1}$$

where

$$S_f(i, M) = \sum_{j=1}^{M-1} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(i), \tag{3.2}$$

and the remainder term is given by

$$R_M = \int_j^k \frac{B_{2M} - B_{2M}(x - [x])}{(2M)!} f^{(2M)}(x)\,dx. \tag{3.3}$$

Here, $B_m(x)$ are the Bernoulli polynomials. They are defined as the coefficients of $z^m/m!$ in the series expansion of $z\exp(xz)/(\exp(z) - 1)$. Explicitly we have

$$\frac{z\exp(xz)}{\exp(z) - 1} = \sum_{m=0}^{\infty} B_m(x)\frac{z^m}{m!}, \quad |z| < 2\pi. \tag{3.4}$$

Furthermore, $B_m = B_m(0)$ are the Bernoulli numbers. They are accordingly given as coefficients of $z^m/m!$ in the series expansion of $z/(\exp(z) - 1)$. Thus,

$$\frac{z}{\exp(z) - 1} = \sum_{m=0}^{\infty} B_m\frac{z^m}{m!}, \quad |z| < 2\pi. \tag{3.5}$$

The first few Bernoulli numbers are $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0$, and $B_4 = -1/30$. All Bernoulli numbers of odd order exceeding

1 are equal to zero. This fact can be used to rewrite (3.5) in the following way:

$$\frac{z}{\exp(z)-1} = 1 - \frac{z}{2} + \sum_{m=1}^{\infty} \frac{B_{2m}}{(2m)!} z^{2m}, \quad |z| < 2\pi. \tag{3.6}$$

We note that the Euler-Maclaurin formula (3.1) expresses the sum $\sum_{n=j}^{k} f(n)$ as the sum of 1) the integral $\int_j^k f(x)\,dx$, 2) the average of the boundary values $f(j)$ and $f(k)$, 3) the difference of two finite sums $S_f(k,M)$ and $S_f(j,M)$, whose terms contain the odd derivatives of the function f, evaluated at the two boundary values k and j, and 4) the remainder term R_M.

We use the Euler-Maclaurin formula to derive the result in (1.11). We start out with putting $f(n) = \varphi(y(n))/\sigma$, and $j = -\infty$. The integral in the Euler-Maclaurin formula is then equal to

$$\int_{-\infty}^{k} f(x)\,dx = \frac{1}{\sigma}\int_{-\infty}^{k} \varphi(y(x))\,dx = \int_{-\infty}^{k} \varphi(y)\,dy = \Phi(y(k)), \tag{3.7}$$

while the average of the boundary values is

$$\frac{1}{2}[f(-\infty) + f(k)] = \frac{1}{2\sigma}\varphi(y(k)). \tag{3.8}$$

To evaluate the sums $S_f(i,M)$ we need the odd derivatives of $f(i)$ with respect to i. Applying (2.4) we get

$$f^{(n)}(i) = \frac{1}{\sigma^{n+1}}\varphi^{(n)}(y(i)) = \frac{(-1)^n}{\sigma^{n+1}} He_n(y(i))\varphi(y(i)). \tag{3.9}$$

Hence we can write

$$S_f(i,M) = -\varphi(y(i))F(z(i),\sigma,M), \tag{3.10}$$

where the function F is defined by

$$F(z,\sigma,M) = \frac{1}{\sigma}\sum_{j=1}^{M-1} \frac{B_{2j}}{(2j)!}\frac{1}{\sigma^{2j-1}} He_{2j-1}(\sigma z). \tag{3.11}$$

By using the explicit expression given in (2.5) for the Hermite polynomial $He_n(y)$ we get

$$F(z,\sigma,M) = \frac{1}{\sigma}\sum_{j=1}^{M-1} \frac{B_{2j}}{(2j)!} \sum_{m=0}^{j-1} \frac{1}{m!}\left(-\frac{1}{2\sigma^2}\right)^m \frac{(2j-1)!}{(2j-1-2m)!} z^{2j-1-2m}. \tag{3.12}$$

Changing the order of summation gives

$$F(z,\sigma,M) = \frac{1}{\sigma}\sum_{m=0}^{M-2} \frac{1}{m!}\left(-\frac{1}{2\sigma^2}\right)^m \sum_{j=m+1}^{M-1} \frac{B_{2j}}{2j}\frac{z^{2j-1-2m}}{(2j-1-2m)!}. \tag{3.13}$$

The sum over j in this expression is related to the derivative of the function G_1 of order $2m$. To show this, we give the power series expansions of the function G_1 and of its even order derivatives. It follows from the definition of the function G_1 in (1.13) and from the power series expansion of the function $z/(\exp(z)-1)$ in (3.6) that

$$G_1(z) = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} z^{2j-1} = \frac{1}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{2j} \frac{z^{2j-1}}{(2j-1)!}, \quad |z| < 2\pi. \tag{3.14}$$

The derivatives of the function G_1 at zero are determined by the Bernoulli numbers as follows:

$$G_1^{(n)}(0) = \frac{B_{n+1}}{n+1}, \quad n = 1, 2, 3, \dots. \tag{3.15}$$

These derivative values are equal to zero if n is even. Furthermore, the derivative of G_1 of order $2m$ is equal to

$$G_1^{(2m)}(z) = \sum_{j=1}^{\infty} \frac{B_{2j+2m}}{2j+2m} \frac{z^{2j-1}}{(2j-1)!} = \sum_{j=m+1}^{\infty} \frac{B_{2j}}{2j} \frac{z^{2j-1-2m}}{(2j-1-2m)!}, \tag{3.16}$$
$$m = 1, 2, 3, \dots, \quad |z| < 2\pi.$$

Thus, the sum over j in (3.13) is equal to the derivative of $G_1(z)$ of order $2m$ if $m \geq 1$ and we put $M = \infty$, and require $|z| < 2\pi$, while it equals $G_1(z) - 1/2$ if $m = 0$. It follows that F can be expressed in terms of G_1 and its even derivatives as follows:

$$F(z, \sigma, \infty) = \frac{1}{\sigma} \left[\sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{1}{2\sigma^2} \right)^m G_1^{(2m)}(z) - \frac{1}{2} \right], \quad |z| < 2\pi. \tag{3.17}$$

We consider now the approximation for the sum $\sum_{n=-\infty}^{k} \varphi(y(n))/\sigma$ that is given by the Euler-Maclaurin formula with the remainder term R_M ignored and with $M = \infty$, and without the restriction that $|z(k)| < 2\pi$. We note that $S_f(-\infty, M) = 0$. By collecting terms we are led to the approximation given by the right-hand side of (1.11). The claim that this provides an asymptotic expansion has not been proved. Numerical support for this claim is given in Section 5.

4. Derivation of the continuity correction approximation

To derive the approximation of the continuity correction in (1.6) we apply the asymptotic expansion in (1.11) on the left-hand side of (1.5) and the modified Maclaurin series in (1.12) on the right-hand side. It follows that the continuity correction $C(k)$ satisfies the relation

$$\sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{1}{2\sigma^2} \right)^m \frac{\partial^{2m}}{\partial z^{2m}} G_2(C(k), z(k)) \tag{4.1}$$
$$\sim 1 - \sum_{m=0}^{\infty} \frac{1}{m!} \left(-\frac{1}{2\sigma^2} \right)^m G_1^{(2m)}(z(k)), \quad \sigma \to \infty.$$

By including only the terms corresponding to $m = 0$ from each of the sums over m we get

$$G_2(C(k), z(k)) = 1 - G_1(z(k)) + \mathrm{O}\left(\frac{1}{\sigma^2}\right), \quad \sigma \to \infty. \tag{4.2}$$

We solve this equation for $C(k)$, using the definitions of the functions G_1 and G_2 in (1.13) and (1.14). If $z(k) = 0$, then we get

$$C(k) = \frac{1}{2} + \mathrm{O}\left(\frac{1}{\sigma^2}\right), \quad z(k) = 0. \tag{4.3}$$

Hence (1.6) holds for $z(k) = 0$. If on the other hand $z(k) \neq 0$, then

$$\exp(-C(k)z(k)) = \frac{z(k)}{\exp(z(k)) - 1} - z(k)\,\mathrm{O}\left(\frac{1}{\sigma^2}\right), \qquad z(k) \neq 0, \quad \sigma \to \infty. \tag{4.4}$$

From this relation we conclude that

$$C(k) = -\frac{1}{z(k)} \log \frac{z(k)}{\exp(z(k)) - 1} + \frac{\exp(z(k)) - 1}{z(k)}\,\mathrm{O}\left(\frac{1}{\sigma^2}\right), \qquad z(k) \neq 0, \quad \sigma \to \infty. \tag{4.5}$$

The result in (1.6) for $z(k) \neq 0$ follows from this, since the first term on the right-hand side is equal to $G(z(k))$, and the second term is $\mathrm{O}(1/\sigma^2)$ if $z(k) = \mathrm{O}(1)$.

The derivation of (1.9) makes use of the first three terms in the series expansion of $G(z)$ about $z = 0$. It is straightforward to show that

$$G(z) \sim \frac{1}{2} + \frac{1}{24}z - \frac{1}{2880}z^3, \quad z \to 0. \tag{4.6}$$

Applying this to (1.6) establishes (1.9).

5. Numerical investigation of the approximation (1.11)

We report here the results of a numerical investigation that has been undertaken to support the claim that the right-hand side of (1.11) actually provides an asymptotic expansion of the left-hand side. We recall that the sum for which an approximation appears in (1.11) is a sum over n-values from $-\infty$ to some finite value k. However, in the numerical study we work with finite sums. For definiteness we choose to work with sums over the n-values from 1 to N, say. We introduce N_1 to denote the corresponding finite sum:

$$\mathrm{N}_1 = \frac{1}{\sigma} \sum_{n=1}^{N} \varphi(y(n)). \tag{5.1}$$

Furthermore, the approximation of the right-hand side of (1.11) that results by replacing the sum over m by the first m_0 terms is denoted

μ	σ	m_0	$\mathrm{E}_1(m_0)$
40	10	1	$2.2 \cdot 10^{-10}$
40	10	4	$9.8 \cdot 10^{-19}$
0	10	1	$3.0 \cdot 10^{-28}$
0	10	4	$9.4 \cdot 10^{-37}$
-40	10	1	$-2.2 \cdot 10^{-10}$
-40	10	4	$-9.8 \cdot 10^{-19}$

TABLE 1. The absolute error $\mathrm{E}_1(m_0)$ committed in approximating the finite sum N_1 is shown for a few values of μ, σ, and m_0 with $N = 100$.

$\mathrm{A}_1(k, m_0)$. It is defined as follows:

$$\mathrm{A}_1(k, m_0) = \Phi(y(k)) + \frac{1}{\sigma}\varphi(y(k)) \left[1 - G_1(z(k)) - \sum_{m=1}^{m_0-1} \mathrm{e}_1(k, m)\right], \tag{5.2}$$

where

$$\mathrm{e}_1(k, m) = \frac{1}{\sigma}\varphi(y(k))\frac{1}{m!}\left(-\frac{1}{2\sigma^2}\right)^m G_1^{(2m)}(z(k)), \quad m \geq 1. \tag{5.3}$$

The sum N_1 is approximated by $\mathrm{A}_1(N, m_0) - \mathrm{A}_1(0, m_0)$, where m_0 is an arbitrary positive integer. The error committed in using this approximation is $\mathrm{E}_1(m_0) = \mathrm{A}_1(N, m_0) - \mathrm{A}_1(0, m_0) - \mathrm{N}_1$.

Table 1 summarizes results of numerical evaluations of the error $\mathrm{E}_1(m_0)$ in the estimate of the finite sum N_1. In all cases we have $N = 100$, and the standard deviation σ of the normally distributed random variable ξ equal to 10. The first two rows of the table deal with the case when the random variable ξ has its mean μ equal to 40. The finite sums over n from 1 to $N = 100$ then start in the left tail and end in the right tail of ξ. The next two rows in the table have $\mu = 0$. Here, the finite sums start in the body and end in the right tail of the random variable ξ. The last two rows in the table have $\mu = -40$. Here, the finite sums are all confined to the right tail of the random variable ξ.

The table shows that the error $\mathrm{E}_1(m_0)$ is small. The high precision achieved by our approximations is exemplified by the values of $\mathrm{E}_1(m_0)$ in case $\mu = 40$. In this case the numerical value of N_1 is approximately equal to 0.99996 11913 97707 09000, with 20 decimals. The approximation achieved by taking $m_0 = 1$ estimates this sum with 9 correct digits, while $m_0 = 4$ improves this to 18 digits!

An even higher precision is observed for the case when $\mu = 0$. In that case we have $y(0) = z(0) = 0$, and it follows that $\mathrm{A}_1(0, m_0) = \Phi(0) + \varphi(0)/(2\sigma)$ is independent of m_0. Thus, errors that vary with

m_0 occur only in $\mathrm{A}_1(N, m_0)$. These errors are small, since they are proportional to $\varphi(y(N))/\sigma$, which is small.

The error estimates $-\mathrm{e}_1(1, m_0)$ agree with high precision with the numerically determined errors in all cases treated in the table. The high precision in the estimates requires a numerical procedure with corresponding high precision. We have used Maple, where numerical computations with arbitrary precision are possible. The Maple procedures and commands used to compute the results in Table 1 are contained in the Maple worksheet appended to this report.

The results in Table 1 give strong support to our claim that the right-hand side of (1.11) gives an asymptotic expansion of the sum in the left-hand side.

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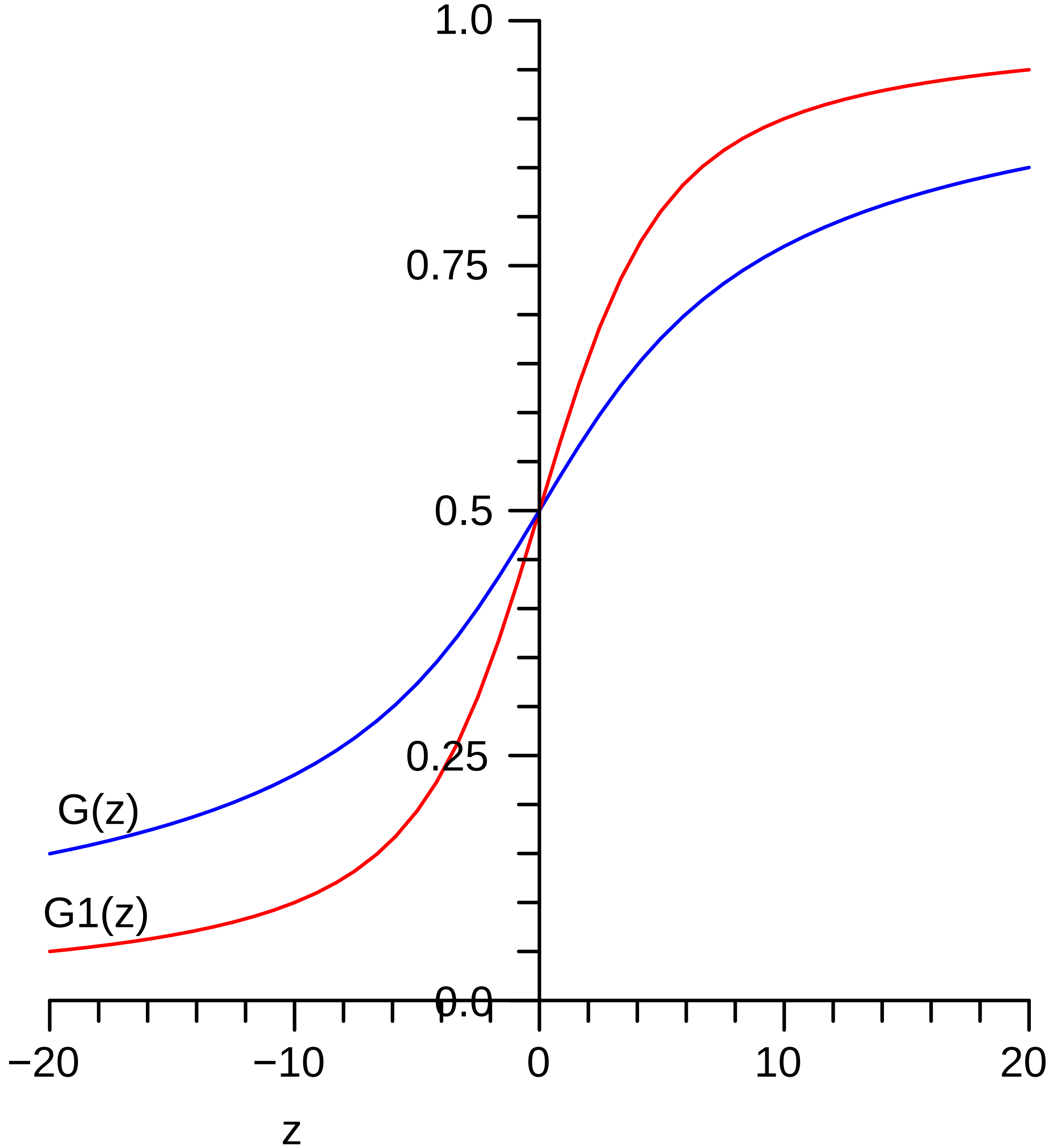

Figure 1. The two functions G and G_1 are plotted as functions of z over the interval from -20 to 20. Both of them can be interpreted as distributions of random variables that possess no moments.

Appendix A. The Maple worksheet for Table 1

This Maple worksheet contains the procedures and commands used to produce the numerical results in Table 1.

```
> restart;
```

Define the normal density and the normal distribution function:

```
> phi:=y->exp(-y^2/2)/sqrt(2*Pi):
  s2:=sqrt(2):
  Phi:=y->(1+erf(y/s2))/2:
```

Define the function G1:

```
> G1:=z->1/(exp(z)-1)-1/z+1:
```

Evaluate the sum from 1 to N numerically:

```
> thesum:=proc(mu,sigma,N)
    local y;
    y:=n->(n-mu)/sigma;
    evalf(add(phi(y(n))/sigma,n=1..N));
  end proc:
```

Determine the approximation A1 of the sum from -infinity to k, including m0 terms of the sum over m. This will be used in the procedure A1error below with the k-values 0 and N.

```
> A1:=proc(mu,sigma,k,m0)
    local yk,zk,a1,a2,a3;
    yk:=(k-mu)/sigma;
    zk:=yk/sigma;
    a1:=evalf(Phi(yk));
    a2:=evalf(phi(yk)/sigma);
    if zk=0 then
      a3:=0.5;
    else
      a3:=1-add((D@@(2*m))(G1)(zk)/m!/(-2*sigma^2)^m,
        m=0..m0-1);
    fi;
    evalf(a1+a2*a3);
  end proc:
```

Determine the error committed in using the approximation A1!

```
> A1error:=proc(mu,sigma,N,m0)
    A1(mu,sigma,N,m0)-A1(mu,sigma,0,m0)
      -thesum(mu,sigma,N);
  end proc:
```

Take the first numerical example. I compute the errors for the m0-values 1,2,3,4.

```
> N:=100:  mu:=40.:  sigma:=10.:  Digits:=25:
  printf("The sum from 1 to N is %22.20f \n",
    thesum(mu,sigma,N));
  printf("Err1 = %0.2E, Err2 = %0.2E, Err3 = %0.2E,
    Err4 = %0.2E \n",seq(A1error(mu,sigma,N,m0),
    m0=1..4));
```

The sum from 1 to N is 0.99996119139770709000
Err1 = 2.21E-10, Err2 = 2.59E-13, Err3 = 4.44E-16,
Err4 = 9.83E-19

Now take mu=0!

```
> N:=100:  mu:=0:  sigma:=10.:  Digits:=40:
  printf("The sum from 1 to N is %22.20f \n",
    thesum(mu,sigma,N));
  printf("Err1 = %0.2E, Err2 = %0.2E, Err3 = %0.2E,
    Err4 = %0.2E \n",seq(A1error(mu,sigma,N,m0),
    m0=1..4));
```

The sum from 1 to N is 0.48005288597992836610
Err1 = 2.97E-28, Err2 = 3.21E-31, Err3 = 4.92E-34,
Err4 = 9.39E-37

The third example:

```
> N:=100:  mu:=-40:  sigma:=10.:  Digits:=25:
  printf("The sum from 1 to N is %0.20E \n",
    thesum(mu,sigma,N));
  printf("Err1 = %0.2E, Err2 = %0.2E, Err3 = %0.2E,
    Err4 = %0.2E \n",seq(A1error(mu,sigma,N,m0),
    m0=1..4));
```

The sum from 1 to N is 2.54248667094519169354E-05
Err1 = -2.20E-10, Err2 = -2.59E-13, Err3 = -4.44E-16,
Err4 = -9.83E-19

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