ON POINCARÉ'S VARIATIONAL PROBLEM IN POTENTIAL THEORY

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Abstract. One of the earliest attempts to rigorously prove the solvability of Dirichlet's boundary value problem was based on seeking the solution in the form of a "potential of double layer", and this leads to an integral equation whose kernel is (in general) both singular and non-symmetric. C. Neumann succeeded with this approach for smoothly bounded convex domains, and H. Poincaré, by a tremendous tour de force, showed how to push through the analysis for domains with sufficiently smooth boundaries but no hypothesis of convexity. In this work he was (according to his own account) guided by consideration of a variational problem involving the partition of energy of an electrostatic field induced by charges placed on the boundary of a domain, more precisely the charge distributions which render stationary the energy of the field inside the domain divided by the energy of the field outside the domain. Unfortunately, a rigorous treatment of this problem was not possible with the tools available at that time (as Poincaré was well aware). So far as we know, the only one to propose a rigorous treatment of Poincarés problem was T. Carleman (in the two dimensional case) in his doctoral dissertation. Thanks to later developments (especially concerning Sobolev spaces, and spectral theory of operators on Hilbert space) one can now give a complete, general and rigorous account of Poincaré's variational problem, and that is the object of the present paper.

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Contents

1. Introduction and historical note 3
2. Prerequisites of potential theory 8
3. Symmetrizable operators 14
4. Poincaré's variational problem 21
5. Schiffer's operator 24
6. Neumann-Poincaré's operator in two dimensions 27
7. Qualitative analysis of the Neumann-Poincaré operator 30
8. Examples 35
8.1. The equilibrium distribution 35
8.2. The ball in $\mathbf{R}^d$ 36
8.3. The ellipse 38
8.4. Lemniscates 39
References 41

1. Introduction and historical note

Riemann built his magnificent function theoretic edifice on the solvability of Dirichlet's boundary value problem. When his proposed proof of this (based on "Dirichlet's principle") was found wanting, it became a priority of the highest order to find a rigorous alternative proof. One early attempt was through potential theory. Since it was soon discovered that it was hopeless to express the solution to the Dirichlet problem in the form of a ("single layer") potential of some suitably selected function defined on the boundary of the domain, emphasis shifted to "double layer" potentials, a very plausible choice insofar as this amounts to replacing the (in general unfindable) Poisson kernel, or normal derivative of the (equally unfindable) Green function, by the normal derivative of the log (in the two dimensional case) which constitutes the singularity of Green's function. It is not our purpose here to review this development, but only to remark that this reduced the Dirichlet problem to an integral equation with "kernel" $K(z, w)$, where z and w are variables ranging over the boundary of the domain in question (the so-called Neumann - Poincaré kernel). A significant feature of this kernel is that, except in the case where the domain is a disk (or ball in $\mathbf{R}^d$) K is not symmetric, so the usual expansion theorems for integral operators were not applicable. Nonetheless, the few examples that were understood indicated that this integral operator had lots of real eigenvalues, as well as (non-orthogonal) eigenfunctions, and there was no general theory available that could explain such behavior. Later, Marty and Korn introduced the notion of symmetrizability of an operator, and showed that this applied to the N - P operator. But, over a decade before this happened Poincaré initiated the study of a certain (self adjoint!) variational problem that did not seem *prima facie* to be related to the N - P kernel, but turns out to be the "high ground" which fully clarifies the "self adjoint features" of the N - P integral operator. So far as we know this fascinating approach has not yet been fully worked out. To do so is our purpose in this paper.

C. Neumann had proven the solvability of Dirichlet's problem in convex domains by a recursive (and in principle constructive) procedure based on calculating an infinite sequence of double layer potentials which were the summands in a series converging to the solution. Later Poincaré had proven by an altogether different method (which he called "balayage") the solvability in domains of quite general character. Thus, in 1897, when Poincaré's paper [34] appeared, there was already a rigorous proof at hand, but Poincaré set himself the task to find an alternate proof based on establishing the convergence of the Neumann series also for nonconvex domains. This he succeeded to do (under fairly strong regularity assumptions: the domain had a C^2 boundary and the boundary values were sufficiently differentiable). Motivation for this undertaking was that the balayage method was not suitable for numerical computation of solutions, and a solution based on Neumann's series was superior in this regard.

Poincaré's new proof was extremely long and technical, and we won't enter into the details here. It is based on energy estimates for the electrostatic field due to charges distributed on the boundary of a domain, more precisely how the energy is partitioned between the part of the field inside the domain and the part lying outside. The final section of the paper has an unusual character: Poincaré poses an extremal problem (more precisely, a sequence of such problems) concerning this partition of energy, which he says guided his steps through the preceding demonstration. But this is placed in a sort of quarantine , and not referred to in the course of the demonstration because, as Poincaré repeatedly tells us, he has been unable to establish the salient details of the extremal problem on a rigorous basis. The paper concludes with the words: *"After having established these results [concerning convergence of the Neumann series] rigorously, I felt obliged in the two final chapters to give an idea of the insights which initially led me to foresee these results. I thought that, despite their lack of rigor, these could be useful as tools for research insofar as I had already used them successfully once."*

Here is what is involved. Consider (we use here, for the most part, Poincaré's notations) a closed surface Γ in $\mathbf{R}^3$ and denote by Ω, Ω' respectively the interior and exterior domains into which $\mathbf{R}^3$ is partitioned by Γ. Let there be given a real valued continuous function u on $\mathbf{R}^3$ and whose restrictions to Ω and Ω' (denoted by W and W') are harmonic functions with finite Dirichlet integrals (denoted $J(W)$ and $J(W')$). Then u is the potential of an electrostatic charge distributed on Γ. (In modern language, this charge is $f := \Delta u$ in the distributional sense, it is a Schwartz distribution with support in Γ.) If we assume the total energy $J(W) + J(W')$ equals 1, what is the minimum value possible for $J(W)$? It is 0, and this is attained if, and only if W is a constant c, and W' is the solution to the Dirichlet problem for the exterior domain with data $W' = c$ on Γ (the "conductor potential" of Γ). Here c is to be chosen so that $J(W') = 1$. The corresponding charge distribution f is (modulo a constant factor) the equilibrium measure for the compact set Γ. All this was well understood at the time. But now Poincaré embarks into *terra incognita*: Consider the analogous extremal problem, but with u conditioned to be orthogonal to the extremal potential for the preceding problem (in the sense that, if u_0 denotes the extremal for that problem, $\int_{\mathbf{R}^3} \nabla u \cdot \nabla u_0 dx = 0$). What is now the minimum for $J(W)/(J(W) + J(W'))$? (We may as well assume the denominator equals 1). Here arises the first of a series of difficulties Poincaré was not able to overcome: is this minimum attained? If so, then denoting by r_1 the minimum and by u_1 some extremal, Poincaré obtains by a routine formal variational procedure the condition that the normal derivative of W_1 equals $-r_1$ times the normal derivative of W_1' at each point of Γ, where W_1 and W_1' are the restrictions of u_1 to Ω, Ω'. (Again there is a lack of rigor insofar as, even assuming existence of an extremal, the existence of the normal derivatives along S is unclear.) Poincaré now proceeds to the next

problem in the succession, whereby u is conditioned to be orthogonal in the indicated sense to u_0 and u_1. And so forth.

The pattern is now clear to a modern observer. There is in the background a Hilbert space H whose entries are pairs of harmonic functions (V, V') defined on Ω, Ω' respectively, the inner product between two such pairs given by

$$\langle (V, V'), (W, W') \rangle = \int_\Omega \nabla V \cdot \nabla W dx + \int_{\Omega'} \nabla V' \cdot \nabla W' dx.$$

(Well, this is not quite accurate since we need some adjustment to rule out elements of the form $V = c, V' = c'$ where c, c' are constants not both 0, yet (V, V') would have norm zero. We'll deal with such technical points later.) In other terms we are in the framework of a Hilbert space H which is the direct sum of the Sobolev spaces $W^{1,2}(\Omega)$ and $W^{1,2}(\Omega')$ and its closed subspace P (the potentials) consisting of those pairs (V, V') whose traces on Γ coincide. We have two Hermitian forms on H:

$$J(V, V') = \int_\Omega |\nabla V|^2 dx$$

and

$$J'(V, V') = \int_{\Omega'} |\nabla V'|^2 dx.$$

The successive minimum problems considered by Poincaré are precisely those employed nowadays (following F. Riesz) in the standard proof of the spectral theorem for compact self-adjoint operators. More precisely, if T is such an operator on a Hilbert space with elements denoted $x, y, ...$ and inner product $\langle ., . \rangle$ we consider the two Hermitian forms $\langle Tx, x \rangle$ and $\langle x, x \rangle$, and begin by (say) seeking the minimum of the former while restricting the latter to be 1. We then repeat the procedure with the competing elements x restricted to be orthogonal to the extremal element x_0 from the first stage, and so forth.

Coming back to Poincaré's problem seen in this light, the immediate question is: Is there a compact linear operator lurking behind the form J? Yes, there is, but we must first replace J by $J - J'$ (which clearly leads to an extremal problem equivalent to the former insofar as the ratios $J/(J + J')$ and $(J - J')/(J + J')$ ...or for that matter J/J' , which Poincaré actually uses...are simply related. It is a highly nontrivial fact that, restricted to the subspace P of H consisting of pairs (V, V') with equal traces on Γ, the form $J - J'$ is completely continuous (to use an older terminology, that held sway when "operators on Hilbert space" were exhibited in terms of Hermitian forms rather than operators). This was first established rigorously by T. Carleman in his remarkable doctoral dissertation [4]. Following modern practice we shall, below, rework all this in the language of operators along the lines of the abstract treatment given by M. G. Krein [20].

This gets us off the ground: extrema in Poincaré's problem are always attained. In terms of the abstract model, we can continue to seek (and find)

the minima of $\langle Tx, x\rangle$ subject to the successively stricter orthogonality constraints imposed on the unit vector x. But one point has to be emphasized: If we assume (as is the case in Poincaré's problem) that $\langle Tx, x\rangle$ takes negative values it attains a minimum on the unit sphere. In general, for any compact operator T, $\langle Tx, x\rangle$ attains a maximum and a minimum on the *unit ball* but only in the case of a positive maximum or a negative minimum can we assert the extremal element has norm 1. Thus, the sequence of minimum problems will continue to furnish an increasing sequence of negative eigenvalues of T so long as $\langle Tx, x\rangle$ attains negative values for x among the remaining (competing) vectors. If this is not so, the process terminates, and either all remaining x are in the kernel of T, or $\langle Tx, x\rangle$ takes positive values for some x. If this is the case, we can maximize $\langle Tx, x\rangle$ among all unit vectors, and then, analogously as before continue to find a decreasing sequence of positive eigenvalues (and associated eigenvectors) of T, which process only terminates if at some point $\langle Tx, x\rangle$ takes no positive values on the eligible set of x. From modern spectral theory we know moreover that to each negative and each positive eigenvalue is associated only a finite dimensional family of eigenvectors, whereas corresponding to the spectral point 0 there may be either no eigenvector, or a family of finite or infinite dimension.

Poincaré seems to have conjectured that (translated into our terminology) an infinite sequence of increasing negative eigenvalues (the first being -1) would exist (i.e. his recursive process would never terminate) and that moreover the associated eigenfunctions would span the Hilbert space. This is a very bold conjecture, implying that the operator associated to the form $J' - J$(which we shall later relate to the so-called Neumann - Poincaré integral operator) has only positive spectrum and moreover is injective. These assertions are true in case Γ is a sphere, but we will show they do not hold generally, indeed not even for ellipsoids of revolution in $\mathbf{R}^3$. For $d = 2$ there are some notable anomalies.

To complete this survey of Poincaré's extremal problem we should take up his variational condition for extrema, which characterizes the extremal potentials $u = (W, W')$ by the condition that the normal derivatives of W and W' , computed along Γ from opposite sides with respect to the same normal vector, are a (negative) constant multiple of one another. We postpone the further examination of this condition, which relates to the aforementioned integral operator and its symmetrization, to a later section.

Let us briefly describe the contents of the paper. Section 2 contains some terminology and conventions plus a collection of known facts from the Newtonian potential theory, seen from the modern point of view of distribution theory and Sobolev spaces. Section 3 is devoted to an abstract symmetrization principle for linear bounded operators acting on a Hilbert space. This theme was popular precisely because of potential theoretic applications, during the first decades of the XX-th century, see for instance [14, 17, 19, 23, 28];

later on, symmetrizable operators have appeared in more abstract studies, for example, of partial differential equations on spaces with two norms, see [9, 25, 38]. We owe to Carleman [4] and Krein [20] the clarification of the subject. We follow their treatment, simplifying whenever possible the reasoning with the help of modern operator theory.

Section 4 presents Poincaré's variational problem in a new light: as the study of the angle operator between two orthogonal decompositions of the space of harmonic fields of finite energy. Roughly speaking, in the presence of a smooth closed surface Γ , this amounts to the decomposition into the fields of single and double layer potential, and respectively that of inner and outer fields. When passing from Euclidean space to Γ, this interpretation, well correlated with the symmetrization scheme, naturally leads to the characteristic values and eigenfunctions of the double layer integral operator (the Neumann-Poincaré operator) in a "negative norm" space defined by the single layer operator. We offer here (as far as we know for the first time for the modern reader) complete proofs of existence, smoothness and completeness of the eigenfunctions appearing in Poincaré's variational problem.

In Section 5 we extend the Hilbert-Beurling transform to act on gradients of harmonic functions in all dimensions and show how one can read the spectrum of the Neumann-Poincaré (boundary) operator, and implicitly that of Poincaré's problem, from this spatial singular integral operator. Section 6 is a brief and novel account of the Beurling operator in its original two real variable form. Without aiming at completeness, we unify and simplify here some classical works in complex analysis revolving around the Fredholm eigenfunctions of a planar domain [1, 2, 3, 40, 41].

Section 7 deals with applications of Poincaré's variational principle. They amply illustrate the flexibility and advantages of his point of view: work on the entire Euclidean space with harmonic fields, and their energy norm, rather than with charges on complicated function spaces supported on the boundary. Specifically, we prove the existence of a domain in $\mathbf{R}^3$ with negative spectrum (of its associated N-P operator), the possibility of "gluing together" finite parts of such disjoint spectra, and analyze the oscillations of certain eigenfunctions of this operator. Section 8 contains examples, comments and open problems. In particular, the ball in $\mathbf{R}^n$ is characterized by the symmetry of its N-P operator.

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2. Prerequisites of potential theory

The aim of this section is to assemble some terminology and basic facts of Newtonian potential theory.

Let Ω be a bounded domain in $\mathbf{R}^d$ with boundary Γ. We assume that Γ is at least C^2-smooth. The $(d-1)$-dimensional surface measure on Γ is denoted by $d\sigma$ and the unit outer normal to a point $y \in \Gamma$ will be denoted n_y.

Throughout this article $E(x,y) = E(x-y)$ denotes the normalized Newtonian kernel:

$$E(x,y) = \begin{cases} \frac{1}{2\pi} \log \frac{1}{|x-y|}, & d = 2, \\ c_d |x-y|^{2-d}, & d \geq 3, \end{cases} \tag{1}$$

where c_d^{-1} is the surface area of the unit sphere in $\mathbf{R}^d$. The signs were chosen so that $\Delta E = -\delta$ (Dirac's delta-function).

For a C^2-smooth function (density) $f(x)$ on Γ we form the fundamental potentials: the *single and double layer potentials* in $\mathbf{R}^d$; denoted S_f and D_f respectively:

$$\begin{aligned} S_f(x) &= \int_\Gamma E(x,y) f(y) d\sigma(y) \\ D_f(x) &= \int_\Gamma \frac{\partial}{\partial n_y} E(x,y) f(y) d\sigma(y). \end{aligned} \tag{2}$$

The surface Γ divides $\mathbf{R}^d$ into two domains $\Omega = \Omega_i$ (interior to Γ) and the exterior Ω_e. Thus the potentials above define pairs of functions (S_f^i, S_f^e) and (D_f^i, D_f^e) which are harmonic in Ω_i and Ω_e respectively.

As is well known from classical potential theory (cf. [15, 42]) denoting by $S_f^i(x)$, $\frac{\partial}{\partial n_x} S_f^i(x)$ (and corresponding symbols with superscript e) the limits at $x \in \Gamma$ from the interior (or exterior), the following relations (known as the jump formulas for the potentials) hold for all $x \in \Gamma$:

$$\begin{aligned} S_f^i(x) &= S_f^e(x); \\ \frac{\partial}{\partial n_x} S_f^i(x) &= \frac{1}{2} f(x) + \int_\Gamma \frac{\partial}{\partial n_x} E(x,y) f(y) d\sigma(y); \\ \frac{\partial}{\partial n_x} D_f^i(x) &= \frac{\partial}{\partial n_x} D_f^e(x); \\ D_f^i(x) &= -\frac{1}{2} f(x) + \int_\Gamma \frac{\partial}{\partial n_y} E(x,y) f(y) d\sigma(y); \\ \frac{\partial}{\partial n_x} S_f^e(x) &= -\frac{1}{2} f(x) + \int_\Gamma \frac{\partial}{\partial n_x} E(x,y) f(y) d\sigma(y); \\ D_f^e(x) &= \frac{1}{2} f(x) + \int_\Gamma \frac{\partial}{\partial n_y} E(x,y) f(y) d\sigma(y). \end{aligned} \tag{3}$$

We warn the reader that different conventions (on the choice of the sign of the fundamental solution or the unit normal) may affect these formulas.

Rather direct computations (see for instance Chapter II in [15], Chapters 18-19 in [42] or [29]) show that the integral kernels

$$K(x,y) := -\frac{\partial}{\partial n_y} E(x-y); \quad K^*(x,y) = -\frac{\partial}{\partial n_x} E(x-y)$$

satisfy on Γ the following estimates, for $d \geq 3$:

$$\begin{aligned} |K(x,y)| &= O(\frac{1}{|x-y|^{d-2}}), \quad x,y \in \Gamma, x \neq y, \\ |K^*(x,y)| &= O(\frac{1}{|x-y|^{d-2}}), \quad x,y \in \Gamma, x \neq y. \end{aligned} \tag{4}$$

For $d = 2$, due to the fact that $\log|z-w|$ is the real part of a complex analytic function $\log(z-w) = \log|z-w| + i\arg(z-w)$, $z, w \in \Gamma$, and by Cauchy-Riemann's equations one obtains

$$K(z,w) = \frac{\partial}{\partial \tau_w} \arg(z-w),$$

where τ_w is the unit tangent vector to the curve Γ. Thus, on any smooth curve $\Gamma \subset \mathbf{R}^2$, the kernels $K(z,w)$ and $K^*(z,w)$ are uniformly bounded.

Returning to the general d-dimensional case, we define on $L^2(\Gamma, d\sigma)$ the *Neumann-Poincaré operator* K:

$$(Kf)(x) = 2\int_\Gamma K(x,y) f(y) d\sigma(y), \quad f \in L^2(\Gamma, d\sigma). \tag{5}$$

The L^2 adjoint K^* will be an integral operator with kernel $K^*(x,y)$. The nature of the diagonal singularity of the kernel $K(x,y)$ shows that K is a compact operator in the Schatten-von Neumann class $\mathcal{C}^p(L^2(\Gamma)), p > d-1$, see [15]. Since the kernel K is bounded when $d = 2$, it is Hilbert-Schmidt on any smooth planar curve. We will show in the next section that K^* is symmetrizable, that is K^* becomes self-adjoint with respect to a different (incomplete) inner product on $L^2(\Gamma)$.

Similarly, the linear operator

$$Sf = S_f|_\Gamma, \quad f \in L^2(\Gamma),$$

turns out to be bounded (from $L^2(\Gamma)$ to the same space). Remark that the representing kernel $E(x,y)$ of S is pointwise non-negative for $d \geq 3$. With these conventions the jump formulas become, as functions on Γ:

$$\begin{aligned} S_f^i &= S_f^e = Sf; \\ \partial_n S_f^i &= \frac{1}{2} f - \frac{1}{2} K^* f; \\ \partial_n S_f^e &= -\frac{1}{2} f - \frac{1}{2} K^* f; \\ D_f^i &= -\frac{1}{2} f - \frac{1}{2} K f; \\ D_f^e &= \frac{1}{2} f - \frac{1}{2} K f. \end{aligned} \tag{6}$$

Above, and always in this paper n designates the *outer* normal to Ω.

In this paper we shall mostly be concerned with the relationship between the spectral analysis of the Neumann-Poincaré operator K and some extremal problems arising from comparing the energies in Ω_i and Ω_e of the single layer potentials defined by densities supported by Γ, cf. the Introduction. To this aim we will use a Sobolev space on $\Omega_i \cup \Omega_e$, its trace on Γ, and some closed subspaces imposed by our considerations.

Let $\mathfrak{H}$ be the space of harmonic functions on $\Omega_i \cup \Omega_e$, vanishing at infinity and with finite energy seminorm:

$$\|h\|_{\mathfrak{H}}^2 = \int_{\Omega_i \cup \Omega_e} |\nabla h|^2 dx. \tag{7}$$

Only locally constant functions are annihilated by this seminorm. It will be necessary to distinguish between the two restrictions of h to the inner and outer domain; we denote $h = (h_i, h_e)$ where $h_i = h|_{\Omega_i}$ and similarly $h_e = h|_{\Omega_e}$. In virtue of Poincaré's inequality the functions h_i and h_e are in the Sobolev $W^{1,2}$-spaces of the corresponding domains. To simplify notation we put henceforth $W^s = W^{s,2}$.

We can regard an element $h \in \mathfrak{H} \subset \mathcal{D}'(\mathbf{R}^d)$ as a distribution defined on the whole space. Then $\Delta h = \mu \in \mathcal{D}'_\Gamma(\mathbf{R}^d)$ (the lower index means $\mathrm{supp}(\mu) \subset \Gamma$) and, by a slight abuse of notation

$$-h(x) = S_\mu(x) = \int_\Gamma E(x,y) d\mu(y), \quad x \in \mathbf{R}^d \setminus \Gamma. \tag{8}$$

If the distribution μ is given by a smooth function times the surface measure of Γ, then $h = S_\mu$ and by (6) $h_i|_\Gamma = h_e|_\Gamma$. Our next aim is to identify the closed subspace of $\mathfrak{H}$ characterized by the latter matching property.

By assumption the surface Γ is smooth. Hence there are linear continuous trace operators

$$\mathrm{Tr} : W^1(\Omega_{i,e}) \longrightarrow W^{1/2}(\Gamma).$$

Moreover, the trace operator from each side of Γ is surjective (and hence it has a continuous right inverse), see for instance [27]. We will denote in short

$$h|_\Gamma = \mathrm{Tr}\ h.$$

If $d \geq 3$, then for any function $f \in W^{1/2}(\Gamma)$ there exist solutions $(h_i, h_e) \in \mathfrak{H}$ to the inner and outer Dirichlet problems with boundary data f: $h_i|_\Gamma = h_e|_\Gamma = f$, see [24].

In the case $d = 2$ the additional assumption $\int_\Gamma f d\sigma = 0$ must be made, to assure the existence of h_e with $h_e(\infty) = 0$ and finite energy, see [24] .

The following consequence of Green's formula will be frequently used in this section. For a harmonic function u in Ω, of class C^2 on the closed domain:

$$2\int_\Gamma u \frac{\partial u}{\partial n} d\sigma = \int_\Omega \Delta(u^2) dx = 2\int_\Omega |\nabla u|^2 dx. \tag{9}$$

Another common form of Green's formula, for arbitrary functions $\phi, \psi \in C^2(\overline{\Omega})$ reads:

$$\int_\Omega \nabla\phi \cdot \nabla\psi dx + \int_\Omega \phi\Delta\psi dx = \int_\Gamma \phi\partial_n\psi d\sigma.$$

As a first application we note an important isometric identification, see [24].

Lemma 2.1. *Let $f \in L^2(\Gamma)$. Then*

$$\langle Sf, f\rangle = \|S_f\|^2_{\mathfrak{H}}. \tag{10}$$

We deduce from here that S is a non-negative self-adjoint operator on $L^2(\Gamma)$. Moreover $Sf = 0$ implies $\nabla S_f = 0$ in $\mathbf{R}^d \setminus \Gamma$, whence S_f is constant on both sides of Γ. But $S_f(\infty) = 0$, so $S_f = 0$ as a distribution on $\mathbf{R}^d$. Therefore $f = -\Delta S_f = 0$. This proves that S is a strictly positive operator on $L^2(\Gamma)$. We will prove below that S is not invertible.

Proposition 2.2. *Assume $d \geq 3$ and let $h = (h_i, h_e) \in \mathfrak{H}$. Then $h_i|_\Gamma = h_e|_\Gamma$ if and only if there exists $\rho \in W^{-1/2}(\Gamma)$ such that $h = S_\rho$.*

Proof. Let H_- be the completion of $L^2(\Gamma)$ with respect to the Hermitian form $\langle Sf, f\rangle = \|\sqrt{S}f\|^2$. Let $H_+ = \mathrm{Ran}\sqrt{S}$, viewed as a non-closed vector subspace of $L^2(\Gamma)$, and also regarded as the domain of the positive unbounded operator $\sqrt{S}^{-1}$. Note that H_+ is a complete space with respect to the norm induced by the form $\langle\sqrt{S}^{-1}f, \sqrt{S}^{-1}f\rangle$. Then the positive operator S can be extended by continuity to an isomorphism $S : H_- \longrightarrow H_+$, and the L^2-pairing

$$\langle\sqrt{S}f, g\rangle = \langle f, \sqrt{S}g\rangle$$

defines a duality between the Hilbert spaces H_+ and H_-.

The above standard duality construction can be correlated to the Dirichlet space seminorm of $\mathfrak{H}$. First we polarize the identity in the Lemma:

$$\langle Sf, g\rangle_{2,\Gamma} = \langle S_f, S_g\rangle_{\mathfrak{H}}, \quad f, g \in L^2(\Gamma).$$

Let $h \in \mathfrak{H}$ be an element with equal traces on Γ:

$$h_i|_\Gamma = h_e|_\Gamma = f \in W^{1/2}(\Gamma).$$

For a C^2-smooth density g on Γ we find via Green's formula:

$$\langle f, g\rangle_{2,\Gamma} = \langle h, S_g\rangle_{\mathfrak{H}}.$$

Thus the linear functional

$$g \mapsto \langle g, f\rangle_{2,\Gamma}$$

is continuous with respect to the seminorm $\langle Sg, g\rangle$ and hence Riesz' lemma implies the existence of an element $k \in L^2(\Gamma)$ such that

$$\langle g, f\rangle_{2,\Gamma} = \langle g, \sqrt{S}k\rangle_{2,\Gamma}, \quad g \in L^2(\Gamma).$$

That is $W^{1/2}(\Gamma) \subset \sqrt{S}L^2(\Gamma) = H_+$.

Conversely, an element $f \in H_+$, can be written as $f = Sk$ with $k \in H_-$. Then the coupling $\langle Sk, k\rangle$ is well defined. Let (k_n) be a sequence in $L^2(\Gamma)$ which converges to k in the topology of H_-. Then

$$\|S_{k_n} - S_{k_m}\|^2_{\mathfrak{H}} = \langle S(k_n - k_m), k_n - k_m\rangle_H.$$

Thus the sequence of potentials (S_{k_n}) is Cauchy in $\mathfrak{H}$ and its limit h satisfies $h_i = Sk = f$, by the continuity of the trace map. Therefore $f \in W^{1/2}(\Gamma)$.

In conclusion $H_+ = W^{1/2}(\Gamma)$, and the $L^2(\Gamma)$ dual of this space is $H_- = W^{-1/2}(\Gamma)$. As noted before, the operator S extends continuously to the space $W^{-1/2}(\Gamma)$, and

$$SW^{-1/2}(\Gamma) = W^{1/2}(\Gamma).$$

In other terms, the density distribution $\rho = \Delta h, h \in \mathfrak{H}$, represents a matching pair $h = (h_i, h_e), h_i|_\Gamma = h_e|_\Gamma$ if and only if $\rho \in W^{-1/2}(\Gamma)$. □

The case $d = 2$ requires again the additional assumption that $\rho(\mathbf{1}) = 0$. Otherwise S_ρ would not have a square summable gradient on the exterior domain.

We define the *subspace of single layer potentials* by

$$\mathfrak{S} = \{h \in \mathfrak{H},\ h_i|_\Gamma = h_e|_\Gamma\}.$$

The orthogonal complement in $\mathfrak{H}$ will be denoted $\mathfrak{D} = \mathfrak{S}^\perp$ and we will identify this with the *space of double layer potentials* belonging to $\mathfrak{H}$.

First we have to define, in a weak sense, the normal derivative of a distribution along Γ. Let $(h_i, h_e), (g_i, g_e)$ be the representatives of elements $h, g \in \mathfrak{H}$. Assume first that both $h_i, g_i \in C^2(\overline{\Omega})$. Then Green's formula yields

$$|\int_\Gamma \frac{\partial h_i}{\partial n} g_i d\sigma| = |\int_\Omega \nabla h_i \cdot \nabla g_i dx| \leq \|\nabla h_i\|_{2,\Omega}\|\nabla g_i\|_{2,\Omega}.$$

By Banach's open mapping theorem, the continuous bijective trace operator

$$\mathrm{Tr} : \{h \in W^1(\Omega), \Delta h = 0\} \longrightarrow W^{1/2}(\Gamma)$$

is bicontinuous, hence, in our situation we find a positive constant C such that

$$\|\nabla g_i\|_{2,\Omega} \leq C\|g_i|_\Gamma\|_{W^{1/2}}.$$

Consequently

$$|\int_\Gamma \frac{\partial h_i}{\partial n} g_i d\sigma| \leq C\|\nabla h_i\|_{2,\Omega}\|g_i\|_{W^{1/2}}.$$

A standard regularization argument shows that every harmonic function g_i in Ω, having finite energy inside Ω (i.e. $\|\nabla g_i\|_{2,\Omega} < \infty$) can be approximated in the energy metric by harmonic functions which are smooth up to the boundary. And we know that the traces g_i exhaust the space $W^{1/2}(\Gamma)$. Thus $\frac{\partial h_i}{\partial n} d\sigma$ defines a linear continuous functional on $W^{1/2}(\Gamma)$, which via

the $L^2(\Gamma)$ duality can be identified with a distribution $\frac{\partial h_i}{\partial n} \in W^{-1/2}(\Gamma)$. Moreover, the above estimate implies

$$\|\frac{\partial h_i}{\partial n}\|_{W^{-1/2}(\Gamma)} \leq C\|\nabla h_i\|_{2,\Omega}.$$

Again by regularization we obtain the following statement.

Proposition 2.3. *The normal derivatives of the boundary values of a pair $(h_i, h_e) \in \mathfrak{H}$ are distributions $\frac{\partial h_i}{\partial n}, \frac{\partial h_e}{\partial n} \in W^{-1/2}(\Gamma)$ depending continuously on $\|h_i\|_{\mathfrak{H}}, \|h_e\|_{\mathfrak{H}}$, respectively, and satisfying the duality identities:*

$$\begin{aligned} \int_\Gamma \frac{\partial h_i}{\partial n} g_i d\sigma = \int_\Gamma h_i \frac{\partial g_i}{\partial n} d\sigma = \int_\Omega \nabla h_i \cdot \nabla g_i dx, \\ \int_\Gamma \frac{\partial h_e}{\partial n} g_e d\sigma = \int_\Gamma g_e \frac{\partial h_e}{\partial n} d\sigma = -\int_\Omega \nabla h_e \cdot \nabla g_e dx, \end{aligned} \tag{11}$$

for every $g = (g_i, g_e) \in \mathfrak{H}$.

We are ready to identify double layer potentials in the space $\mathfrak{H}$.

Corollary 2.4. *Let $h = (h_i, h_e) \in \mathfrak{H}$. The following conditions are equivalent:*

a) $h \in \mathfrak{D} (= \mathfrak{S}^\perp)$;
b) $\frac{\partial h_i}{\partial n} = \frac{\partial h_e}{\partial n}$ (in $W^{-1/2}(\Gamma)$);
c) There exists $f \in W^{1/2}(\Gamma)$ such that $h = D_f$.
In this case $f = h_e - h_i$.

Proof. Assume that $h, g \in \mathfrak{H}$ are orthogonal elements. Then the above proposition yields

$$\int_\Gamma (\frac{\partial h_i}{\partial n} g_i - \frac{\partial h_e}{\partial n} g_e) d\sigma = 0.$$

Any element $f \in W^{1/2}(\Gamma)$ can be realized as $f = g_i = g_e$ for a proper choice of g, hence b) follows. Conversely, if b) holds, then the same identity implies a).

Assume that b) is true and define $f = h_e - h_i$. Then D_f is well defined and satisfies by (6) $(D_f)_e - (Df)_i = f$, $\frac{\partial (D_f)_i}{\partial n} = \frac{\partial (D_f)_e}{\partial n}$. This proves that the pair of harmonic functions $h - D_f$ form a single harmonic function on $\mathbf{R}^d$ which vanishes at infinity, and hence identically. Therefore $h = D_f$. □

The only elements of $\mathfrak{H}$ annihilated by the energy seminorm are scalar multiples of $(\mathbf{1}, 0)$. This is the double layer potential of the constant function, and is, therefore, orthogonal to $\mathfrak{S}$. By the boundary formula $(D_f)_e = \frac{1}{2} f - \frac{1}{2} K$ we infer $K\mathbf{1} = \mathbf{1}$.

Another distinguished element of $\mathfrak{H}$ is provided by the equilibrium distribution ρ on Γ; namely $(1, h) \in \mathfrak{S}$, that is $S\rho = 1$ and $h = S^e_\rho$, see Example 1.

The following result is known as *Plemelj's symmetrization principle*, see [33, 16, 18] .

Lemma 2.5. *The operators $S, K : L^2(\Gamma) \longrightarrow L^2(\Gamma)$ satisfy the identity*

$$KS = SK^*. \tag{12}$$

Proof. Let $f \in C^2(\Gamma)$. It is sufficient to prove

$$D_{Sf}(x) = S_{\partial_n Sf}(x),\ x \in \Omega_e.$$

Indeed, passing from x to a point x_0 of the boundary we would obtain

$$Sf(x_0) - KSf(x_0) = -SK^* f(x_0) + Sf(x_0)$$

and the proof would be complete.

The asserted identity follows again from Green's fromula:

$$D_{Sf}(x) = \int_\Gamma \frac{\partial}{\partial n_y} E(x,y)[\int_\Gamma E(y,z) f(z) d\sigma(z)] =$$

$$\int_\Gamma f(z)[\int_\Gamma \frac{\partial}{\partial n_y} E(x,y) E(z,y) d\sigma(y)] d\sigma(z) =$$

$$\int_\Gamma f(z)[\int_\Gamma E(x,y) \frac{\partial}{\partial n_y} E(z,y) d\sigma(y)] d\sigma(z) =$$

$$\int_\Gamma E(x,y)[\frac{\partial}{\partial n_y} \int_\Gamma f(z) E(z,y) d\sigma(z)] d\sigma(y) = S_{\partial_n Sf}(x).$$

□

3. Symmetrizable operators

In the present section we closely follow Carleman ([4], §11) and isolate an abstract symmetrization principle for linear bounded operators. A few decades after Carleman, the same symmetrization technique was analyzed in detail, in a very flexible general framework, by M.G. Krein [20].

To be more specific, we are studying a bounded non-selfadjoint operator acting on a Hilbert space H which becomes symmetric with respect to a another, bounded scalar product on H. Prior works related to the same idea have appeared early in the study of the Neumann-Poincaré operator. A good account of these works is contained in the survey article by Hellinger and Toeplitz [14].

The theorem below is aimed at and will be directly applicable to the Neumann-Poincaré operator.

Let H be an infinite dimensional, separable, complex Hilbert space and let $\mathcal{C}_p = \mathcal{C}_p(H), p \geq 1$, be the Schatten-von Neumann class of compact operators acting on H, see [12].

Theorem 3.1. *Let $p \geq 1$ and let $M \in \mathcal{C}_p(H)$ be a linear bounded operator with the property that there exists a strictly positive bounded operator R such that $R^2 M$ is self-adjoint.*

Then the spectrum of M is real and for every non-zero eigenvalue λ, if $(M-\lambda)^m f = 0$ for some $m > 1$, then $(M-\lambda)f = 0$.

Moreover, the eigenvectors of M^*, including the null vectors, span H.

By R strictly positive we mean $R \geq 0$ and $\ker R = 0$.

Proof. Let us define on H the bounded sesquilinear form:

$$\langle f, g\rangle_R = \langle Rf, Rg\rangle, \quad f, g \in H.$$

Note that the space H is complete with respect to this new norm if and only if the operator R is invertible.

The assumption in the statement implies

$$\langle Mf, g\rangle_R = \langle RMf, Rg\rangle = \langle Rf, RMg\rangle = \langle f, Mg\rangle_R.$$

That is M is a symmetric operator with respect to the new scalar product.

We prove a little more than the statement. Namely that there exists a bounded self-adjoint operator $A \in \mathcal{C}_p$ with the property:

$$AR = RM. \tag{13}$$

Let N denote the self-adjoint operator:

$$N = R^2M = M^*R^2.$$

We regularize the inverse of R by a small positive parameter ϵ; to the effect that the strong operator topology limit $(R+\epsilon)^{-1}R \to I$ exists when ϵ tends to zero. And for any operator $L \in \mathcal{C}_q$ the limit $(R+\epsilon)^{-1}RL \to L$ exists in the norm topology of $\mathcal{C}_q$ (by a finite rank approximation argument).

Fix a positive integer n so that $p < 2n$ and consider the operator:

$$A_\epsilon = (R+\epsilon)^{-1}R^2M(R+\epsilon)^{-1}.$$

Then $A_\epsilon \in \mathcal{C}_p \subset \mathcal{C}_{2n}$, and

$$|A_\epsilon|^2 = A_\epsilon^2 = (R+\epsilon)^{-1}R^2M(R+\epsilon)^{-2}R^2M(R+\epsilon)^{-1}.$$

In virtue of the cyclic invariance of the trace we obtain:

$$\mathrm{tr}|A_\epsilon|^{2n} = \mathrm{tr}[(R+\epsilon)^{-2}R^2M]^{2n} \to \mathrm{tr}M^{2n} < \infty.$$

Thus the family of operators $(A_\epsilon)_{\epsilon>0}$ is bounded in $\mathcal{C}_{2n}$, hence relatively compact in the weak topology of the same ideal.

On the other hand, $A_\epsilon R$ converges in the norm topology of $\mathcal{C}_{2n}$ to RM. This implies that any weak limit point A of $(A_\epsilon)_{\epsilon>0}$ must satisfy the identity $AR = RM$. Since the operator R was assumed to be injective, all limit points coincide with a uniquely determined operator $A \in \mathcal{C}_{2n}$ satisfying identity (1).

Moreover,

$$RAR = R^2M = M^*R^2 = N. \tag{14}$$

Since R is injective with dense range, it follows from

$$\langle ARf, Rg\rangle = \langle Rf, ARg\rangle, \quad f, g \in H,$$

that A is self-adjoint.

It remains to prove that $A \in \mathcal{C}_p$. Let λ_k be a non-zero value in the spectrum of A and let f_k be a corresponding eigenvector, normalized by the condition $\|f_k\| = 1$. Then, by (13),

$$M^* R f_k = R A f_k = R \lambda_k f_k.$$

Hence $\lambda_k \in \sigma(M^*)$. Since $M^* \in \mathcal{C}_p$, the convergence of the series

$$\sum_{k=1}^{\infty} |\lambda_k|^p < \infty$$

shows that $A \in \mathcal{C}_p$. In the enumeration λ_k we allow multiplicities and can assume by the compactness of A that $|\lambda_k| \geq |\lambda_{k+1}|$, for all values of k.

As for the zero eigenvectors h of A, they span its kernel, and each element Rh is annihilated by M^*, as follows from the identity $M^* R h = R A h = 0$. Since the vectors $(f_k)_{k\geq 1}$ span, together with $h \in \ker A$, the space H, the M^*-eigenvectors $(Rf_k)_{k\geq 1}$ and $(Rh)_{Ah=0}$ span H by the density of the range of R. This proves the last part of the theorem.

Remark next that the spectral decomposition of A provides a norm convergent series:

$$Af = \sum_k \lambda_k \langle f, f_k \rangle f_k, \quad f \in H. \tag{15}$$

In general, if $E(\mu, T)$ denotes the spectral projection of an isolated point in the spectrum of a linear bounded operator T, then:

$$E(\mu, T)^* = E(\overline{\mu}, T^*),$$

see [6] Theorem VII.3.10. Hence the spectral spaces $H_\lambda(M), H_\lambda(M^*)$ of M and M^* corresponding to the same isolated eigenvalue λ have the same dimension. Now, due to the intertwining relations $AR = RM$, $RA = M^* R$, the operator R induces injective maps:

$$H_\lambda(M) \xrightarrow{R} H_\lambda(A) \xrightarrow{R} H_\lambda(M^*),$$

for all $\lambda \neq 0$. Since $\dim H_\lambda(M) = \dim H_\lambda(M^*)$ we deduce that the three spectral spaces above are isomorphic.

Furthermore, let $E(\lambda, M)H$ be the spectral subspace of M corresponding to the eigenvalue $\lambda \in \sigma(M) \setminus \{0\}$. By compactness, $E(\lambda, M)H$ is a finite dimensional space, and in general it may contain generalized eigenvectors of M, that is solutions g of the equation $(M - \lambda)^m g = 0$. But by hypothesis, $M|_{E(\lambda,M)H}$ is symmetric with respect to the new scalar product induced by R, and therefore diagonalizable on that subspace. Hence $E(\lambda, M)H$ only contains eigenvectors of M.

□

We expand below a couple of comments and corollaries derived from the preceding proof.

1. *Eigenfunction expansions.* Let $g_k \in H$ be the eigenvector of M corresponding to the non-zero eigenvalue λ_k. In case of higher multiplicities, we repeat λ_k accordingly in the enumeration of the spectrum. Set $f_k = Rg_k$, and normalize g_k so that $\|f_k\| = 1$. The spectral decomposition of A implies the norm convergent expansion

$$RMf = \sum_k \lambda_k \langle f, R^2 g_k \rangle Rg_k, \quad f \in H,$$

or equivalently, the $\|.\|_R$-convergent series:

$$Mf = \sum_k \lambda_k \langle f, g_k \rangle_R \; g_k.$$

By applying another R we obtain

$$M^* R^2 f = \sum_k \lambda_k \langle R^2 f, g_k \rangle R^2 g_k.$$

In general, however, the stronger continuity assumption

$$\|Mf\| \leq C\|Rf\|, \quad f \in H,$$

(with C a positive constant) is needed for the series

$$Mf = \sum_k \lambda_k \langle f, R^2 g_k \rangle g_k, \quad f \in H,$$

to be norm convergent. See for details [20].

Along the same lines, note that the self-adjoint operator $N = RAR$ admits a non-orthogonal, norm convergent decomposition into rank-one self-adjoint operators (cf. (13)):

$$Nh = \sum_k \lambda_k \langle h, Rf_k \rangle Rf_k, \quad h \in H. \tag{16}$$

It does not follow from (16) that the eigevectors of M are complete in H.

2. *Compactness and eigenvalues in the negative space.* Let $\tilde{H}$ be the completion of H in the $\|.\|_R$-norm, and let $\tilde{M}$ be the linear continuous extension of M there. We call $\tilde{H}$ the negative space by analogy with distribution theory (and the theory of Gelfand triples).

Our next aim is to prove that $\tilde{M}$ is compact and self-adjoint on $\tilde{H}$ and that every eigenvector of $\tilde{M}$ corresponding to a non-zero eigenvalue belongs to H, and hence it is an eigenvector of M.

The intertwining identity $AR = RM$ implies that $\tilde{M}$ is compact on $\tilde{H}$ (even in the same class $\mathcal{C}_p$). We prove the compactness of $\tilde{M}$. Let (ϕ_n) be a bounded sequence in $\tilde{H}$. We can find a sequence (h_n) in H such that $\|R(\phi_n - h_n)\| \leq 2^{-n}$, $n \geq 1$. Then by the compactness of A, the sequence (ARh_n) has a convergent subsequence which we will denote by the same symbols (ARh_n). But $ARh_n = RMh_n$. That is the sequence Mh_n is

convergent in the $\|.\|_R$-norm. Thus the sequence $(\tilde{M}h_n)$ is convergent in $\tilde{H}$, and so is the sequence $(\tilde{M}\phi_n)$.

We extend next the operator R to $\tilde{H}$. Let $\tilde{h} \in \tilde{H}$ and consider a sequence (h_n) in H which converges in $\tilde{H}$ to $\tilde{h}$. By the very definition of the negative norm, the sequence (Rh_n) is Cauchy in H. Define $\tilde{R}h = \lim_n Rh_n$. Moreover, this definition implies that the operator $\tilde{R} : \tilde{H} \longrightarrow H$ is continuous in the respective norms, and

$$\|\tilde{R}\tilde{h}\| \leq \|\tilde{h}\|_R.$$

Since the range of R is dense in H, it follows that $\tilde{R}\tilde{H} = H$.

On the other hand, we can consider $R^{-1} : RH \longrightarrow H$ as an unbounded self-adjoint operator. Its domain, $H_+ = RH$ is complete with respect to the graph norm $\|h\|_{H_+} = \|R^{-1}h\|$. Let L be a linear continuous functional on H_+. By the Riesz representation lemma, and by the preceding definition of $\tilde{R}$ there exists an element $g = \tilde{R}\tilde{g} \in H$ such that

$$L(Rh) = \langle h, g\rangle = \langle h, \tilde{R}\tilde{g}\rangle = \langle Rh, \tilde{h}\rangle.$$

Therefore, the scalar product of H defines a non-degenerate continuous pairing between the Hilbert spaces $H_+ = RH$ and $\tilde{H}$. We claim that $\tilde{M}$ is the adjoint of M with respect to this duality pairing. Indeed, let $\tilde{h} \in \tilde{H}$ and $f \in H_+$ be arbitrary elements. The above definitions imply:

$$\langle \tilde{M}\tilde{h}, f\rangle = \langle \tilde{R}\tilde{M}\tilde{h}, R^{-1}f\rangle =$$

$$\langle A\tilde{R}\tilde{h}, R^{-1}f\rangle = \langle \tilde{R}\tilde{h}, AR^{-1}f\rangle =$$

$$\langle \tilde{h}, RAR^{-1}f\rangle = \langle \tilde{h}, M^*f\rangle.$$

Consequently, by general duality theory for the spectral spaces, the spectrum of $\tilde{M}$ is real, equal to the spectrum of the compact operator $M^* : H_+ \longrightarrow H_+$, and the multiplicities of the non-zero eigenvalues are equal. But every eigenvalue of $M^* : H_+ \longrightarrow H_+$ is an eigenvalue of $M^* : H \longrightarrow H$. And conversely, we have proved that every eigenvalue of M^* is of the form $R^2 g_k$, where $Mg_k = \lambda_k g_k$. Hence the spectral subspace corresponding to a non-zero eigenvalue of $M^* : H \longrightarrow H$ is included in H_+, and has the same multiplicity as the spectral subspace of M. But the operator M is a restriction of $\tilde{M}$ to H and as such coincides with the latter on all finite dimensional subspaces of H. Therefore, all eigenvectors of $\tilde{M}$ corresponding to a non-zero eigenvalue belong to H, and coincide with the eigenvectors of M.

For more details about the above proof, and another way of reaching the same conclusion, the reader may consult [20].

3. *Min-max.* Having understood the full spectral picture of the operator M and its continuous extension $\tilde{M}$ we are now prepared to discuss the meaning of the abstract variational principle which makes the subject of the present article.

We know that $\tilde{M} : \tilde{H} \longrightarrow \tilde{H}$ is a compact operator, with the same real spectrum as M, and the same eigenfunctions: $\tilde{M}g = \lambda g,\ \lambda \neq 0,\ g \in H$. The spectrum can have positive and negative eigenvalues, with only 0 as an accumulation point. We denote the positive eigenvalues by:

$$\lambda_0^+ \geq \lambda_1^+ \geq \lambda_2^+ \geq \ldots \geq 0,$$

and, similarly,

$$\lambda_1^- \leq \lambda_2^- \leq \ldots \leq 0.$$

The associated eigenfunctions are correspondingly denoted by g_k^+, g_k^-. The asymmetry in the notation comes from the particular integral operator we deal with in this paper.

Both sequences might be finite (in case M is a finite rank operator), or one can be finite and the other infinite, etc.

The classical Courant-Fischer minimax principle yields:

$$\min_V \max_{f \in V \setminus \{0\}} \frac{\langle \tilde{M}f, f\rangle_R}{\|f\|_R} = \lambda_k^+,$$

where $V \subset \tilde{H}$ is a subspace of codimension k. Moreover, the minimum is attained on the subspace $V = \{g_0^+, \ldots, g_{k-1}^+\}^\perp \subset \tilde{H}$.

Note that in this process we obtain a non-increasing sequence of optimal values which converges, or stabilizes to 0. Due to the compactness of $\tilde{M}$, the min-max process will never detect the negative eigenvalues.

On the other hand, keeping the same notational conventions (in case of finite negative spectrum) we obtain:

$$\max_V \min_{f \in V \setminus \{0\}} \frac{\langle \tilde{M}f, f\rangle_R}{\|f\|_R} = \lambda_k^-,$$

where $V \subset \tilde{H}$ is a subspace of codimension $k-1$. Again the optimal subspace is generated by the vectors which are orthogonal to $g_1^-, \ldots, g_{k-1}^-$. For the same reason this max-min process will not reach non-negative eigenvalues.

The next Proposition is obtained by assembling a part of the above observations. As a step towards unifying the notation used in the rest of the paper we put $S = R^2$.

Proposition 3.2. *Let $M \in \mathcal{C}_p(H),\ p \geq 1$, be a linear operator satsifying the identity $M^*S = SM$, where S is a positive bounded operator on H. Let $\lambda_0^+ \geq \lambda_1^+ \geq \ldots \geq 0 \geq \ldots \geq \lambda_2^- \geq \lambda_1^-$ be the eigenvalues of M repeated according to their multiplicity, and let $g_k^+, g_k^- \in H$ be the corresponding eigenvalues.*

Then,

$$\lambda_k^+ = \max_{f \perp \{g_0^+, \ldots, g_{k-1}^+\}} \frac{\langle SMf, f\rangle}{\langle Sf, f\rangle}. \tag{17}$$

and similarly

$$\lambda_k^- = \min_{f \perp \{g_1^-, \dots, g_{k-1}^-\}} \frac{\langle SMf, f\rangle}{\langle Sf, f\rangle}. \tag{18}$$

4. *Operators with a continuous kernel.* A slightly stronger assumption than the S-symmetry of the compact operator M discussed above is the factorization $M = LS$, where $S > 0$ and $L \geq 0$ is a compact operator. Indeed, $SM = SLS = M^*S$.

This is the class of symmetrizable operators with a continuous kernel, in the terminology of Krein [20]. We put as before $R = \sqrt{S}$. Since $M = (LR)R$, we find

$$\|Mf\| \leq \|LR\| \|f\|_R, \quad f \in H.$$

Therefore the continuous extension $\tilde{M}$ maps continuously and compactly the negative space $\tilde{H}$ into H.

Henceforth we assume that $S > 0$ and $L \geq 0$. As a consequence of the compactness and positivity of L one obtains the convergence in H of the series:

$$Lf = \sum_k \lambda_k \langle f, g_k\rangle g_k, \quad f \in H, \tag{19}$$

see Theorem 9 of [20]. Indeed, recall that $(g_k)_k$ is an orthonormalized system of eigenvectors in $\tilde{H}$, which spans together with $\ker \tilde{M}$ the whole space. In particular, for a vector $f \in H$ we have the convergent Fourier series in $\tilde{H}$:

$$f = \sum_k \langle f, Sg_k\rangle g_k + \sum_j \langle f, \xi_j\rangle_{\tilde{H}} \xi_j,$$

where (ξ_j) is a completion of (g_k), with vectors in $\ker \tilde{M}$, to an orthonormal basis. By applying the operator M to the above sum we find

$$LSf = Mf = \sum_k \lambda_k \langle Sf, g_k\rangle g_k,$$

where the convergence is now assured in H. Let

$$L_N f = \sum_{k \leq N} \lambda_k \langle Sf, g_k\rangle g_k,$$

so that

$$\langle LSf, Sf\rangle = \sum_k \lambda_k |\langle Sf, g_k\rangle|^2 \geq$$

$$\sum_{k \leq N} \lambda_k |\langle Sf, g_k\rangle|^2 = \langle L_N Sf, Sf\rangle, \quad f \in H.$$

But the range of S is dense in H, so that $L_N \leq L$, as self-adjoint operators. Then it is well known that $L' = \mathrm{SOT} - \lim_N L_N$ exists, it is a bounded operator and moreover $L(Sf) = L'(Sf)$ for all f. In conclusion $L = L'$ and the convergence of the expansion (19) is proved.

Assume in addition that the operator L is strictly positive. Then the system $(g_k) \subset H$ of eigenfunctions of M spans H, and at the same time it is an orthonormal basis in $\tilde{H}$. Indeed, if $\langle x, g_k\rangle = 0$ for all k, then $Lx = 0$ and $x = 0$.

The expansion (19) can be regarded as an abstract analogue of Mercer's theorem in the theory of integral operators, see [20].

5. *The norm of a symmetrizable operator.* Let $L(H)$ denote the C^*-algebra of linear bounded operators acting on the Hilbert space H. Assuming $R^2M = M^*R^2$ as in Theorem 3.1 we immediately obtain the formula

$$\|M\|_{L(\tilde{H})} = \lambda_0^+. \tag{20}$$

Indeed,using the notation introduced in the proof of Theorem 3.1,

$$\langle Mf, f\rangle_{\tilde{H}} = \langle ARf, Rf\rangle_H \le \lambda_0^+ \|f\|_{\tilde{H}}^2 = \lambda_0^+ \|Rf\|_H^2,$$

and the inequality is attained by the compactness of the operator A.

For a symmetrizable operator M as before, the following non-trivial norm estimate holds:

$$\|M\|_{L(\tilde{H})} \le \|M\|_{L(H)},$$

see [20].

The applications envisaged in this article are more natural in the context of *real* Hilbert spaces. However, all operators arising in potential theory have real kernels, and, accordingly, all the results discussed in the present section apply to them. To see this formally, the reader should consider an antilinear isometric involution J on H (complex conjugation on function spaces, or on the coefficients of a distinguished orthonormal basis) and assume that the operators M, S, R, in our notation are real, that is they commute with J.

4. Poincaré's variational problem

We keep the notation and conventions introduced in the preliminaries: $\mathfrak{H}$ is the space of pairs of harmonic functions (h_i, h_e) defined on Ω_i, respectively Ω_e, $h_e(\infty) = 0$, and having finite energy.

The prehilbertian space $\mathfrak{H}$ possesses two natural direct sum decompositions:

$$\mathfrak{H} = \mathfrak{S} \oplus \mathfrak{D} = \mathfrak{H}_i \oplus \mathfrak{H}_e.$$

By definition, the latter subspaces are

$$\mathfrak{H}_i = \{(h_i, 0) \in \mathfrak{H}\}, \quad \mathfrak{H}_e = \{(0, h_e) \in \mathfrak{H}\}.$$

Let P_s, P_d, P_i, P_e be the corresponding orthogonal projections. The only subspace $\mathfrak{N} = \mathbf{C}(\mathbf{1}, 0)$ annihilated by the seminorm satisfies:

$$\mathfrak{N} \subset \mathfrak{D} \cap \mathfrak{H}_i.$$

Sometimes we will prefer to work within a Hilbert space, and then we will replace tacitly $\mathfrak{H}$ by $\mathfrak{H} \ominus \mathfrak{N}$.

Recall that the boundary single layer potential S is an L^2-positive operator mapping $W^{-1/2}(\Gamma)$ onto $W^{1/2}(\Gamma)$. The L^2 pairing between the two Sobolev spaces is still denoted $\langle S\rho, f\rangle_{2,\Gamma},\ \rho \in W^{-1/2}(\Gamma), f \in W^{1/2}(\Gamma)$.

We will prove that the quadratic form used by Poincaré in his variational problem is essentially the "angle operator" between these orthogonal decompositions (that is $P_sP_iP_s$ or an affine combination such as $P_s(P_e - P_i)P_s$).

Lemma 4.1. *Let $g \in W^{-1/2}(\Gamma)$ (assuming $g(\mathbf{1}) = 0$ in case $d = 2$). Then*

$$\frac{\langle (P_e - P_i)S_g, S_g\rangle_{\mathfrak{H}}}{\|S_g\|^2_{\mathfrak{H}}} = \frac{\langle KSg, g\rangle_{2,\Gamma}}{\langle Sg, g\rangle^2_{2,\Gamma}}. \tag{21}$$

Proof. Due to the continuity of all terms we can assume that g is a smooth function on Γ. By Green's formula, the jump formulas (6) and the opposite orientation of Γ with respect to the exterior domain we find

$$\int_{\Omega_e} |\nabla S_g|^2 dx = -\int_\Gamma Sg\ \partial_n S^e_g d\sigma = \langle Sg, \frac{1}{2}g + \frac{1}{2}K^*g\rangle_{2,\Gamma},$$

and

$$\int_{\Omega_i} |\nabla S_g|^2 dx = \int_\Gamma Sg\ \partial_n S^i_g d\sigma = \langle Sg, \frac{1}{2}g - \frac{1}{2}K^*g\rangle_{2,\Gamma}.$$

Therefore,

$$\langle (P_e - P_i)S_g, S_g\rangle_{\mathfrak{H}} = \langle KSg, g\rangle_{2,\Gamma},$$
$$\|S_g\|^2_{\mathfrak{H}} = \langle Sg, g\rangle_{2,\Gamma}.$$

□

In view of Plemelj's symmetrization principle (Lemma 2.5), the conditions of the abstract symmetrization scheme in Theorem 3.1 are met for the second Rayleigh quotient above. Accordingly we can state the following theorem, whose main points were foreseen by Poincaré.

Theorem 4.2. *Let $\Omega \subset \mathbf{R}^d$ be a bounded domain with smooth boundary Γ and let $\Omega_e = \mathbf{R}^d \setminus \overline{\Omega}$. Let S_ρ denote the single layer potential of a distribution $\rho \in W^{-1/2}(\Gamma)$, $(\rho(\mathbf{1}) = 0$ in case $d = 2)$.*

Define successively, as long as the maximum is positive, the energy quotients

$$\lambda^+_k = \max_{\rho \perp \{\rho^+_0, \dots, \rho^+_{k-1}\}} \frac{\|\nabla S_\rho\|^2_{2,\Omega_e} - \|\nabla S_\rho\|^2_{2,\Omega}}{\|\nabla S_\rho\|^2_{2,\mathbf{R}^d}}, \tag{22}$$

where the orthogonality is understood with respect to the total energy norm. The maximum is attained at a smooth distribution $\rho^+_k \in W^{1/2}(\Gamma)$.

Similarly, define

$$\lambda^-_k = \min_{\rho \perp \{\rho^-_1, \dots, \rho^-_{k-1}\}} \frac{\|\nabla S_\rho\|^2_{2,\Omega_e} - \|\nabla S_\rho\|^2_{2,\Omega}}{\|\nabla S_\rho\|^2_{2,\mathbf{R}^d}}. \tag{23}$$

The minimum is attained at a smooth distribution $\rho^-_k \in W^{1/2}(\Gamma)$.

The potentials $S_{\rho_k^\pm}$ together with all $S_\chi \in \ker K (\chi \in W^{-1/2}(\Gamma))$, are mutually orthogonal and complete in the space of all single layer potentials of finite energy.

The stronger than expected regularity of the eigenfunctions ($\rho_k^\pm \in W^{1/2}(\Gamma)$) was explained in abstract form, in the last section. The equilibrium distribution of Ω provides the first function ρ_0^+ in this process: $S\rho_0^+ = 1, S_{\rho_0^+}|_\Omega = 1$. The first eigenvalue is always $\lambda_0^+ = 1$ and has multiplicity equal to one (cf. Example 8.1).

Lemma 4.1 gives a precise correlation between the above Poincaré variational problem and the Neumann-Poincaré operator.

Corollary 4.3. *The spectrum of the Neumann-Poincaré operator K, multiplicities included, coincides with the spectrum $(\lambda_k^\pm)$ of Poincaré's variational problem, together with possibly the point zero. The extremal distributions for the Poincaré problem are exactly the eigenfunctions of K.*

In practice it is hard to work directly with the N-P operator on $L^2(\Gamma)$. Instead, the following interpretation of the extremal solutions to Poincaré's variational problem is simpler and more flexible. This also goes back to Poincaré's memoir [34] , and it was constantly present in the works of potential theory in the first decades of the twentieth century, cf. for instance [33].

Let us start with an eigenfunction $f \in L^2(\Gamma)$ of the operator K^*. Then

$$K^* f = \lambda f \ \Rightarrow KSf = SK^* f = \lambda Sf,$$

and by the jump formulas (6)

$$\partial_n S_f^i = \frac{1-\lambda}{2} f, \quad \partial_n S_f^e = \frac{-1-\lambda}{2} f.$$

The associated energies are

$$J_i[f] = \int_{\Omega_i} |\nabla S_f|^2 dx = \frac{1-\lambda}{2} \langle Sf, f\rangle,$$

$$J_e[f] = \int_{\Omega_e} |\nabla S_f|^2 dx = \frac{1+\lambda}{2} \langle Sf, f\rangle.$$

To verify our computations, simply note that

$$\frac{J_e[f] - J_i[f]}{J_e[f] + J_i[f]} = \lambda.$$

The characteristic feature of the above single layer potential S_f is encoded in the following statement.

Proposition 4.4. *A pair of harmonic functions $(h_i, h_e) \in \mathfrak{H}$ represents an extremal potential for Poincaré's variational problem (distinct from the equilibrium distribution) if and only if, there are non-zero constants α, β such that*

$$h_i|_\Gamma = \alpha h_e|_\Gamma \quad \partial_n h_i|_\Gamma = \beta \partial_n h_e|_\Gamma. \tag{24}$$

Proof. For the proof we simply change h_e into αh_e, and remark that this is a single layer potential of a charge ρ. The second proportionality condition implies, via the jump formulas, $K^*S\rho = \lambda S\rho$ for a suitable λ. By the injectivity of S we find $K\rho = \lambda\rho$, and the general symmetrization framework implies $\rho \in W^{1/2}(\Gamma)$. □

Note that in the above proof we could as well renormalize the normal derivatives and assume that $(\beta h_i, h_e)$ is a double layer potential of a density $f \in W^{1/2}(\Gamma)$ which turns out to be an eigenfunction of the N-P operator K.

5. Schiffer's operator

The operator angle, in the Hilbert space $\mathfrak{H}$ of potentials, between the subspace of single layer potentials and that of elements supported by Ω can also be computed as $P_i(P_d - P_e)P_i$. In dimension two, a recapturing of the latter as a singular integral operator, was studied by M. Schiffer and S. Bergman [3, 40, 41]. We present below a general d-dimensional construction of an integral operator acting on harmonic fields as $P_i(P_d - P_e)P_i$. This justifies the title of the section.

Our first aim is to link, in arbitrary dimension, $P_i(P_d - P_e)P_i$ to a boundary operator.

Let $f \in W^{1/2}(\Gamma)$ and $g \in W^{-1/2}(\Gamma)$, and assume that

$$D_f^e + S_g^e = 0.$$

By passing to boundary values,

$$\frac{1}{2}f - \frac{1}{2}Kf + Sg = 0. \tag{25}$$

In other terms

$$Sg \in \operatorname{ran}(I - K) = \ker(I - K^*)^{\perp}.$$

But we know that the density g_0 of the equilibrium distribution satisfies $Sg_0 = \mathbf{1}$, the constants are the only elements in $\ker(I - K)$ and that $(I - K)S = S(I - K^*)$. Whence $\ker(I - K^*) = \mathbf{C}g_0$. Thus equation (25) has a solution if and only if $\langle g, \mathbf{1}\rangle = \int g d\sigma = 0$. Then we can write $f = 2(K - I)^{-1}Sg$, based on the observation that $(I - K) : \mathbf{1}^{\perp} \longrightarrow \operatorname{ran}(I - K)$ is an invertible operator. By these orthogonality assumptions we can assume without loss of generality that both functions f, g have real values. Then, using one more time the assumption $D_f^e + S_g^e = 0$, the jump formulas (6) and (25) we find

$$\langle (P_d - P_s)(D_f + S_g), D_f + S_g\rangle_{\mathfrak{H}} = \int_\Omega \nabla(D_f - S_g)\cdot\nabla(D_f + S_g)dx =$$

$$\int_\Gamma (D_f^i - S_g^i)\partial_n(D_f^i + S_g^i)d\sigma = \int_\Gamma (-\frac{1}{2}f - \frac{1}{2}Kf - Sg)\partial_n(D_f^i + S_g^i - D_f^e - S_g^e)d\sigma =$$

$$-\int_\Gamma Kf g d\sigma.$$

Similarly,

$$\|D_f + S_g\|^2_{\mathfrak{H}} = \int_\Omega |\nabla(D_f + S_g)|^2 dx = -\int_\Gamma f g d\sigma.$$

We have arrived at the following isometric identification.

Lemma 5.1. *Let $h \in \mathfrak{H}$ be an element supported by the inner domain, i.e. $h_e = 0$. Decompose $h = D_f + S_g$, $f \in W^{1/2}(\Gamma), g \in W^{-1/2}(\Gamma)$. Then $g, f \perp \mathbf{1}$, $(I-K)^{-1}Sg$ is well defined, and*

$$\frac{\langle (P_d - P_s)h, h\rangle_{\mathfrak{H}}}{\|h\|^2_{\mathfrak{H}}} = \frac{\langle K(I-K)^{-1}Sg, g\rangle_{2,\Gamma}}{\langle (I-K)^{-1}Sg, g\rangle_{2,\Gamma}}. \tag{26}$$

To put this into the abstract symmetrization scheme we have only to replace $L^2(\Gamma)$ by the codimension one subspace $H = \mathbf{1}^\perp$ of vectors orthogonal to the constants. The operator $(I-K)^{-1}S$ is strictly positive on H, and can replace S in Proposition 3.2. However, in general, the operator K does not leave H invariant. To correct this we consider the orthogonal projection P_H of K onto H and the compression $K_1 = P_H K P_H$ of K; then

$$K_1(I-K)^{-1}S = (I-K)^{-1}SK_1^*.$$

Indeed, start with $f, g \in H$ satisfying $f = (I-K)^{-1}Sg$, that is $(I-K)f = Sg$. Then $(I-K)Kf = KSg = SK^*g$, or equivalently $(I-K)K_1 f = SK_1^* g$, which is the relation to be proved.

The following analogue of Poincaré's principle holds.

Theorem 5.2. *Let $\Omega \subset \mathbf{R}^d$ be a bounded domain with smooth boundary. Let P_s, P_d denote the orthogonal projections of the energy space $\mathfrak{H}$ onto the subspace of single, respectively double layer potentials. Let $\mathfrak{H}_i$ be the subspace of functions vanishing on the complement of Ω.*

Define successively, as long as the maximum is positive, the energy quotients

$$\lambda_k^+ = \max_{\substack{h\in\mathfrak{H}_i \\ h\perp\{h_1^+,\dots,h_{k-1}^+\}}} \frac{\langle (P_d - P_s)h, h\rangle_{\mathfrak{H}_i}}{\|h\|^2_{\mathfrak{H}_i}}. \tag{27}$$

Then the maximum is attained at an element $h_k^+ \in \mathfrak{H}_i$.

Similarly, define

$$\lambda_k^- = \min_{\substack{h\in\mathfrak{H}_i \\ h\perp\{h_1^-,\dots,h_{k-1}^-\}}} \frac{\langle (P_d - P_s)h, h\rangle_{\mathfrak{H}_i}}{\|h\|^2_{\mathfrak{H}_i}}. \tag{28}$$

The minimum is attained at $h_k^- \in \mathfrak{H}_i$.

Exactly as before, the link to the operator K is very simple.

Corollary 5.3. *In the conditions of the Theorem, the spectrum of the Neumann-Poincaré operator K consists, including mutiplicities, of $\{\lambda_k^\pm;\ k \geq 1\}$ together with the points $\{0, 1\}$.*

Remark that the eigenvalue $\lambda_0^+ = 1$ cannot be detected by the above variational scheme. This is due to the fact that the corresponding eigenfunction $\mathbf{1}$ of K cannot satisfy the compatibility condition $\frac{-1}{2}\mathbf{1} + \frac{1}{2}K\mathbf{1} - Sg = 0$, (which would mean $g = 0$). This scenario would produce the pair $(\mathbf{1}, 0)$ of zero total energy.

The extremal solutions to the above problem are precisely

$$h_k^{\pm} = S_{u_k^{\pm}} + D_{g_k^{\pm}},$$

where

$$Kg_k^{\pm} = \lambda_k^{\pm} g_k^{\pm},$$

and

$$2Su_k^{\pm} = (1 - \lambda_k^{\pm})g_k^{\pm}.$$

Next we describe a realization of the abstract angle operator $P_i(P_d - P_s)P_i$. To this aim we consider an arbitrary element $h \in \mathfrak{H}$ and its *harmonic field* $\nabla h \in L^2(\mathbf{R}^d, dx)$. We define, for points $x \in \Omega$

$$\Pi(\nabla h)(x) = p.v.\nabla_x \int_{\mathbf{R}^d} \nabla_y E(x,y) \cdot \nabla_y h dy. \tag{29}$$

Lemma 5.4. *The operator* Π *acts as follows:*

$$\Pi(\nabla S_g)(x) = -\nabla S_g(x), \quad \Pi(\nabla D_f)(x) = 0,$$

whenever $x \in \Omega,\ f \in W^{1/2}(\Gamma),\ g \in W^{-1/2}(\Gamma)$.

Proof. Let $x \in \Omega$ and choose $\epsilon > 0$ small enough so that the closed ball $\overline{B_\epsilon(x)}$ is contained in Ω. Denote by $\Omega^\epsilon = \Omega \setminus \overline{B_\epsilon(x)}$. Let f, g be densities as in the statement. By a repeated use of Green's formula and Gauss' mean value theorem on the sphere $|x - y| = \epsilon$ we obtain:

$$\int_{\Omega^\epsilon \cup \Omega_e} \nabla_y E(x,y) \cdot \nabla_y S_g(y) dy =$$

$$-\int_{\Omega_e} S_g(y) \Delta E(x-y) dy - \int_\Gamma S_g(y)(\partial_n^e - \partial_n^i) E(x,y) d\sigma(y)$$

$$-\int_{\Omega^\epsilon} S_g(y) \Delta E(x-y) dy - \int_{|x-y|=\epsilon} S_g(y) \partial_n E(x-y) d\sigma(y) =$$

$$-S_g(x).$$

Thus $\Pi(\nabla S_g)(x) = -\nabla S_g(x)$.

We proceed similarly for double layer potentials:

$$\int_{\Omega^\epsilon \cup \Omega_e} \nabla_y E(x,y) \cdot \nabla_y D_f(y) dy =$$

$$-\int_\Gamma E(x,y) \partial_n^e D_f(y) d\sigma(y) + \int_\Gamma E(x,y) \partial_n^i D_f(y) d\sigma(y)$$

$$+\int_{|x-y|=\epsilon} E(x-y) \partial_n D_f(y) d\sigma(y) =$$

$$E(\epsilon)\int_{|x-y|=\epsilon}\partial_n D_f(y)d\sigma(y)=0.$$

□

In order to properly define our angle operator on harmonic fields we consider the analogue of the Bergman space in Ω:

$$\mathfrak{B}(\Omega)=\{\nabla u;\ \Delta u=0\ \text{ in }\ \Omega, \|\nabla u\|_{2,\Omega}<\infty\}.$$

It is a Hilbert space of vector valued functions. We also define

$$T_\Omega : \mathfrak{B}(\Omega)\longrightarrow\mathfrak{B}(\Omega),$$

by

$$T_\Omega(\nabla u)(x)=(I+2\Pi)(\nabla u,0)(x),\ x\in\Omega.$$

Recall that every element $(u,0)\in\mathfrak{H}$ can be expressed as $(u,0)=D_f+S_g$. Thus, for such a pair

$$T_\Omega\nabla(D_f+S_g)=\nabla(D_f-S_g)\in\mathfrak{B}(\Omega).$$

Moreover, the computations at the beginning of this section yield the following result.

Proposition 5.5. *The linear operator $T_\Omega : \mathfrak{B}(\Omega)\longrightarrow\mathfrak{B}(\Omega)$ is compact and its spectrum coincides with the spectrum of the Neumann-Poincaré operator, with the exception of the point* 1.

Proof. For the proof we have only to observe the validity of the identity

$$\frac{\langle T_\Omega(\nabla u),\nabla u\rangle}{\|\nabla u\|^2}=\frac{\langle K_1(I-K)^{-1}Sg,g\rangle_{2,\Gamma}}{\langle(I-K)^{-1}Sg,g\rangle_{2,\Gamma}},$$

where $(u,0)=D_f+S_g$.

The point 1 is missing from the spectrum of T_Ω because $(u,0)=S_g+D_f$ imples $f,g\in H$, that is $f,g\perp\mathbf{1}$ and the compression of K to the space H eliminates the point 1 from the spectrum, cf. the text following (26).

□

Since the operator T_Ω is invariant under homotheties $x\mapsto tx,\ t>0$, we deduce that the spectrum of the Neumann-Poincaré operator associated to a domain Ω is invariant under all shape preserving transformations (i.e., translations, rotations and homotheties) of Ω.

6. Neumann-Poincaré's operator in two dimensions

The natural connection to complex analytic functions provides a better understanding of the spectral analysis of the Neumann-Poincaré operator in two dimensions. There are a few specific two dimensional phenomena, whose discovery goes back to the works of Ahlfors [1] , Bergman [2, 3], Plemelj [33], Schiffer [40, 41] and Springer [43]. This section is devoted to the proofs of some of the specifically two dimensional results which are related to the main theme of the present article. We do not aim at completeness, and for

example we do not discuss the link between the eigenvalues of the Neumann-Poincaré operator and quasiconformal mappings. These and other results are well exposed in the aforementioned works. On the other side none of these papers emphasizes on the relationship between the spectrum of the N-P operator and Poincaré's extremum problem, the focus of the present study.

We return to the notation introduced in the preliminaries, with some specific adaptations to dimension two: Γ is a C^2-smooth Jordan curve, surrounding the domain $\Omega \subset \mathbf{C}$, and having Ω_e as exterior domain. We denote by $z, w, \zeta, ...$ the complex coordinate in $\mathbf{C}$ and by $\partial_{\overline{z}} = \frac{\partial}{\partial \overline{z}}$ the Cauchy Riemann operator, and so on. The area measure will be denoted dA. The space $\mathfrak{H}$ consists of (real-valued) harmonic functions h on $\mathbf{C} \setminus \Gamma$ having square summable gradients:

$$h \in \mathfrak{H} \Leftrightarrow \int_{\Omega \cup \Omega_e} |\ \partial_{\overline{z}} h(z)|^2 dA(z) < \infty,\ h(\infty) = 0.$$

Note that the gradients $\partial_{\overline{z}} h$ are now square summable complex anti-analytic functions. In other terms, in our notation $\mathfrak{B}(\Omega)$ is the complex conjugate of the Bergman space $A^2(\Omega)$ of Ω.

The single and double layer potentials are in this case strongly related to Cauchy's integral. For instance,

$$(Kf)(z) = \int_\Gamma f(\zeta) \mathrm{Re}\ [\frac{d\zeta}{2\pi i(\zeta - z)}] = \frac{1}{2\pi} \int_\Gamma f(\zeta)\ d\ \arg(\zeta - z),$$

see [15].

The following complex antilinear singular integral operator plays the role of the symmetry $P_d - P_s$ in our notation. Let $F = \nabla S_f,\ f \in W^{1/2}(\Gamma)$, be regarded as a single anti-analytic function defined on all $\Omega \cup \Omega_e$. Define the Hilbert (sometimes called Beurling) transform

$$(TF)(z) = \mathrm{p.v.} \frac{1}{\pi} \int_{\Omega \cup \Omega_e} \frac{\overline{F(\zeta)}}{(\overline{\zeta} - \overline{z})^2} dA(\zeta) \tag{30}$$

Lemma 6.1. *Let $h \in \mathfrak{H}$ be represented as $h = D_f + S_g,\ f \in W^{1/2}(\Gamma), g \in W^{-1/2}(\Gamma)$. Then*

$$T\nabla(D_f + S_g) = \nabla(D_f - S_g). \tag{31}$$

The proof is very similar to the proof of Lemma 5.4 and we omit it. In other terms, returning to our old notation:

$$T\nabla h = \nabla(P_d - P_s)h, \quad h \in \mathfrak{H}.$$

In particular we note the following simple but important fact.

Corollary 6.2. *The antilinear transform T is an isometric isomorphism of the space $\mathfrak{B}(\Omega) \oplus \mathfrak{B}(\Omega_e)$ onto itself.*

By repeating the definitions of the last section we set

$$T_\Omega : \mathfrak{B}(\Omega) \longrightarrow \mathfrak{B}(\Omega), \quad T_\Omega(F)(z) = T(F, 0)(z), \quad z \in \Omega,$$

where $(F,0)$ means the extension of $F \in \overline{A^2(\Omega)}$ by zero on Ω_e. Thus the operator T_Ω and the one described above coincide as linear transformations over the real field. Consequently we have proved the following result. For convenience we state it for analytic functions, rather than their complex conjugates. The correspondence between one space and another is an obvious isometry.

Theorem 6.3. *Let Ω be a bounded planar domain with C^2 smooth boundary and let $T_\Omega : A^2(\Omega) \longrightarrow A^2(\Omega)$ be the antilinear operator*

$$[T_\Omega f](z) = \text{p.v.}\frac{1}{\pi}\int_\Omega \frac{\overline{f(\zeta)}}{(\zeta - z)^2} dA(\zeta), \quad f \in A^2(\Omega),\ z \in \Omega.$$

Then T_Ω is compact and the eigenvalues of the antilinear eigenvalue problem

$$T_\Omega f_k = \lambda_k f_k$$

coincide (multiplicities included) with the spectrum of the Neumann-Poincaré operator, except the eigenvalue 1. *The eigenfunctions (f_k) are orthogonal and complete in $A^2(\Omega)$.*

As a matter of fact we know more, namely the isometric identification

$$\frac{\langle T_\Omega(\nabla u), \nabla u\rangle_\Omega}{\|\nabla u\|^2_\Omega} = \frac{\langle K_1(I-K)^{-1}Sg, g\rangle_{2,\Gamma}}{\langle (I-K)^{-1}Sg, g\rangle_{2,\Gamma}},$$

where this time we work with antianalytic functions $\nabla(u)$, with $u \in \mathfrak{H}$. The reader can easily transform this into an identity for the associated operator acting on the Bergman space.

This, and the symmetrization construction yield

$$\|T_\Omega\| = \lambda_1^+ \tag{32}$$

where λ_1^+ is the largest eigenvalue of K less than 1.

Note the ambiguity of phase in the eigenvalue problem $T_\Omega f = \lambda f$. By multiplying f by a complex number τ of modulus one, the complex antilinearity of T_Ω implies $T_\Omega f = \tau^2 \lambda f$. On the other hand, we have identified T with an $\mathbf{R}$-linear operator $(P_d - P_s)$ acting on gradients of real harmonic functions. This simple observation leads to the following characteristic symmetry of the Neumann-Poincaré operator specific for two variables.

Proposition 6.4. *Let $\Gamma \subset \mathbf{R}^2$ be a C^2-smooth Jordan curve. Then, except the point* 1, *the spectrum of the Neumann-Poincaré operator acting on $L^2(\Gamma)$ is symmetric with respect to the origin, multiplicities included: $\lambda \in \sigma(K), \lambda < 1$ if and only if $-\lambda \in \sigma(K)$.*

Proof. Let $\lambda \in \sigma(K) \setminus \{1\}$ and let $(u,0) \in \mathfrak{H}$ be the associated eigenfunction of the operator $P_i(P_d - P_e)P_i$, cf. Proposition 5.5. By the above correspondence there exists an anti-analytic function $F = \partial_{\bar z} u$ satisfying $T_\Omega F = \lambda F$. Let $G = iF$ and remark that the antilinearity of T_Ω implies $T_\Omega G = -\lambda G$. Remark also that $G = \partial_{\bar z}\tilde u$, where $\tilde u$ is the harmonic conjugate of u. Thus, the eigenvector in $\mathfrak{H}$ corresponding to the eigenvalue $-\lambda$ is simply $(\tilde u, 0)$. □

The eigenvalue 1 does not have a companion, see Example 8.1.

Another symmetry is available from the above framework.

Proposition 6.5. *Let Ω be a bounded planar domain with C^2-smooth boundary and let Ω_e be the exterior domain. Then the Bergman space operators T_Ω and T_{Ω_e} have equal spectra.*

Proof. Let $(F, 0)$ be an eigenvector of T_Ω, corresponding to the eigenvalue λ. Denote $T(F, 0) = (\lambda F, G)$. Since $T^2 = I$ we get $(F, 0) = \lambda T(F, 0) + T(0, G) = (\lambda^2 F, \lambda G) + T(0, G)$. Thus $T(0, G) = ((1 - \lambda^2)F, -\lambda G)$. This means $-\lambda \in \sigma(T_{\Omega_e})$ and by the preceding symmetry principle $\lambda \in \sigma(T_{\Omega_e})$. □

7. Qualitative analysis of the Neumann-Poincaré operator

The present section is devoted to a few aspects of spectral analysis of the Neumann-Poincaré operator, obtained via Poincaré's variational principles.

Our first aim is to prove the existence of a domain in $\mathbf{R}^3$ which carries a negative spectrum (of the associated N-P operator), arbitrarily close to -1. This infirms Poincaré's guess, based on the case of a ball, that the spectrum of spatial bodies is always non-negative. We start by constructing a couple of cut-off functions.

Lemma 7.1. *For every positive δ there exists an odd C^∞-function $\psi : \mathbf{R} \longrightarrow \mathbf{R}$, such that*

$$\psi(t) = t, \quad t \in [-1, 1],$$

$$\int_{|t|>1} |\psi'(t)|^2 dt < \delta,$$

$$\lim_{\pm t \to \infty} \psi(t) = \pm 1.$$

Proof. Let Λ be a positive constant. Choose, for $t > 1$

$$\psi(t) = 1 + (t - 1)e^{-\Lambda^2 (t-1)^2}.$$

Then

$$\psi'(t) = e^{-\Lambda^2 (t-1)^2}[1 - 2(t - 1)^2 \Lambda^2],$$

and the conditions in the statement are met for Λ sufficiently large. □

Lemma 7.2. *Given $r > 1$ there exists a function $\phi : \mathbf{R} \longrightarrow [0, \infty)$, such that*

$$\phi(x) = \begin{cases} 1 & |x| \le 1, \\ 0 & |x| \ge r^2 \end{cases}$$

and

$$\int_{-\infty}^{\infty} \phi'(x)^2 dx = \frac{C}{r^2 - 1},$$

for a universal constant C.

Proof. Let χ be a smooth function satisfying $\chi(x) = 1$ for $x < 0$ and $\chi(x) = 0$ for $x \geq 1$. Define

$$\phi(x) = \begin{cases} 1 & |x| \leq 1, \\ \chi(|\frac{x-1}{r^2-1}|), & |x| \geq 1. \end{cases}$$

Then

$$\phi'(x) = \frac{1}{r^2-1}\chi'(\frac{x-1}{r^2-1}), \quad x > 1,$$

and similary for $x < -1$. A change of variable in the integral will prove the statement. □

Theorem 7.3. *There exists a bounded domain with smooth boundary such that its Neumann-Poincaré operator has negtive spectrum, arbitrarily close to* -1.

Proof. Fix a small positive ϵ and consider the cylinder

$$G = \{(x, y, z);\ x^2 + y^2 < 1,\ |z| < \epsilon\}.$$

We will approximate G arbitrarily close from inside by a domain Ω with smooth boundary. For instance Ω can be an ellipsoid of revolution. The distribution u which will produce a negative spectrum is given on the boundary Γ of Ω by the function

$$u(x, y, z) = \phi(x^2 + y^2)\psi(z/\epsilon),$$

where the functions ϕ, ψ are those constructed in the previous lemmas, with parameters to be determined in the course of the proof.

Note that for $(x, y, z) \in \Omega$ the function $u(x, y, z) = z/\epsilon$ is harmonic. Thus

$$J[u] = \int_\Omega |\nabla u|^2 dxdydz.$$

Since the domain Ω was chosen close enough to the cylinder G, the energy $J[u]$ can be made arbitrarily close to

$$J[u] \approx \int_G |\nabla(u)|^2 dxdydz = \pi \int_{-\epsilon}^{\epsilon} \frac{1}{\epsilon}|\psi'(\frac{z}{\epsilon})|^2 d(\frac{z}{\epsilon}) = \frac{2\pi}{\epsilon}.$$

By abuse of notation we will not carry below this approximation.

To estimate the energy of the field outside the domain, although the function u is not harmonic there, in virtue of Dirichlet's principle

$$J'[u] := J_e[u] \leq \int_{\Omega_e} |\nabla(u)|^2 dxdydz \approx \int_{G_e} |\nabla(u)|^2 dxdydz.$$

It will be simpler to estimate the total energy, starting from the particular form of the function u:

$$J[u] + J'[u] \leq 2\pi \int_1^r 4\rho^2 \phi'(\rho^2)^2 \rho d\rho \ . \ \int_{-\infty}^{\infty} |\psi(\frac{z}{\epsilon})|^2 dz +$$

$$2\pi \int_0^r \phi(\rho^2)^2 \rho d\rho \ . \ \int_{-\infty}^{\infty} \frac{1}{\epsilon}|\psi'(\frac{z}{\epsilon})|^2 d(\frac{z}{\epsilon}).$$

Next we denote by $C > 0$ a generic universal constant. According to our lemmas,

$$J[u] + J'[u] \leq \frac{C}{r^2-1} + \pi r^2 \left\{ \int_{-\epsilon}^{\epsilon} + \int_{|z|>\epsilon} \right\} \frac{1}{\epsilon} |\psi'(\frac{z}{\epsilon})|^2 d(\frac{z}{\epsilon}) \leq$$

$$\frac{C}{r^2-1} + \pi r^2 \frac{2}{\epsilon} + \pi r^2 \frac{\delta}{\epsilon} =$$

$$\frac{C}{r^2-1} + \pi (r^2-1)\frac{2}{\epsilon} + J[u] + \pi r^2 \frac{\delta}{\epsilon}.$$

We choose $\delta = \sqrt{\epsilon},\ r^2 - 1 = \sqrt{\epsilon}$, and obtain

$$J[u] + J'[u] \leq J[u] + \frac{C}{\sqrt{\epsilon}},$$

and finally

$$J'[u] \leq \frac{C}{\sqrt{\epsilon}} = J[u] C\sqrt{\epsilon}.$$

This proves that the distribution u on the boundary of Ω produces an arbitrarily small energy ratio $J'[u]/J[u]$. According to Poincaré's principle, the associated N-P operator has then a point in the spectrum arbitrarily close to the value -1. □

The above proof has a plausible physical interpretation: If a condenser consisting of two neighboring parallel plates is charged by placing large charges of equal magnitude and opposite signs on the plates, most of the energy of the resulting field is in the space between the plates.

The next result complements the previous example.

Theorem 7.4. *Let Ω be a domain with smooth boundary Γ in $\mathbf{R}^3$. Then, there exists a positive constant c such that, if λ is an eigenvalue of the Neumann-Poincaré operator associated to Ω with $\lambda < -1 + c$ and f a corresponding eigenfunction, f takes both positive and negative values on $\partial\Omega$. The constant c can be chosen uniformly for all domains Ω with C^2 boundary having uniformly bounded principal curvatures.*

Proof. We use $c_1, c_2, ...$ to denote positive numerical constants. The proof is by contradiction. Suppose then that f is a non-negative eigenfunction associated to eigenvalue $\lambda < -1 + c$. We'll show that for small c this leads to a contradiction. We may assume w.l.o.g. that

$$\int_\Gamma f d\sigma = 1. \tag{33}$$

We first show

$$J[f] \leq c_1. \tag{34}$$

Indeed, we have $\lambda f(x) = \int_\Gamma K(x,y) f(y) d\sigma(y)$, where K denotes the Neumann - Poincaré kernel. By iteration we get

$$\lambda^3 f(x) = \int_\Gamma K_3(x,y) f(y) d\sigma(y) \tag{35}$$

where K_3 is the third iterate of K. It is known that $|K_3(x,y)| \leq c_2$ on $\Gamma \times \Gamma$ (see [15]). Hence, in view of (35), and since $|\lambda|$ cannot be small due to our assumption,

$$f(x) \leq c_3, \quad x \in \Gamma. \tag{36}$$

This immediately implies (34) .

To proceed with the proof, note that we may (and do) assume Ω is contained in the ball $B_{1/2}$ (where B_R denotes the ball of radius R centered at 0). Let F be the function on ∂B_1 such that the measure $Fd\sigma$ is the balayage of $fd\sigma$. Then, $P(x,y)$ denoting Poisson's kernel for the unit ball (where x is in B_1 and y in ∂B_1),

$$F(y) = \int_\Gamma P(x,y) f(x) d\sigma(x).$$

Since $P(x,y) \geq c_4$ for $|x| \leq 1/2$ (and hence for x in $\partial\Omega$) this implies

$$F(y) \geq c_4, \quad y \in \partial B_1. \tag{37}$$

Let h denote the function identically equal to 1 on ∂B_1. It is important to stress that h is a multiple of the equilibrium potential and $J(h) > 0$. Since $F \geq c_4 h$, we have $J[F] \geq (c_4)^2 I[h]$ (the energy functional is monotonic for positive charges) , and so

$$J[F] \geq c_5. \tag{38}$$

Now, from (38) follows that, for the electrostatic field in $\mathbf{R}^3$ engendered by F, the part in the exterior of B_1 has energy greater than $(1/2)c_5$ (this is a consequence of the theory of Poincaré's variational problem for the ball, as presented in Section 8.2). Since F arises from f via balayage, this is identical (outside B_1) with the field due to f , and so, *a fortiori* , the energy $J_e[f]$ of the field due to f outside Ω is not less than $(1/2)c_5 =: c_6$. Recall now from (34) that $J[f] \leq c_1$. Since

$$\lambda = \frac{J_e[f] - J_i[f]}{J_e[f] + J_i[f]} = 2\frac{J_e[f]}{J[f]} - 1 \geq 2(c_6/c_1) - 1 = c_7 - 1,$$

whereas we assumed $\lambda < -1 + c$. This is a contradiction for $c = c_7$, and the proof is concluded. □

Remark. By a similar argument one can show e.g. if f, g are eigenfunctions each associated to some eigenvalue in the range $(-1, -1+c)$, then every nontrivial linear combination of f and g changes sign, and the same is true for more eigenfunctions.

Our next goal is by applying some standard approximation theory to the N-P operator, to "glue" two domains into a one, without distorting too much finitely many points of the union of the two spectra. The precise statement follows.

Theorem 7.5. *Let $\Omega_1, \Omega_2 \subset \mathbf{R}^d$ be two bounded domains with smooth boundary. Let $F_i \subset \sigma(K_i) \setminus \{0\}, i = 1, 2$, be two disjoint finite sets in the spectra of the corresponding N-P operators.*

For a given $\epsilon > 0$, there exists a bounded domain Ω with smooth boundary and associated N-P operator K such that for every $\lambda \in F_1 \cup F_2$,

$$\operatorname{dist}(\lambda, \sigma(K)) < \epsilon.$$

If the point λ has multiplicity m, then the ball $B(\lambda, \epsilon)$ contains exactly m points of $\sigma(K)$, counting multiplicities.

Proof. We will base our reasoning on the following known fact: If $T \in L(H)$ is a linear bounded operator, acting on a Hilbert space, and λ is an isolated point of its spectrum of finite multiplicity m, then for every small $\epsilon > 0$ there exists a positive δ such that, whenever an operator $S \in L(H)$ satisfies $\|S - T\| < \delta$, the spectrum of S contains exactly m points (counting multiplicities) in the disk centered at λ and having radius ϵ. A possible proof can be derived from counting the poles of the resolvent $(z - S)^{-1}$ along the fixed contour $|z - \lambda| < \epsilon$, see for instance [12].

We start with the two domains $\Omega_{1,2}$ and the given finite sets of spectral points $F_{1,2}$. We know that a translation of the domain will not change the spectrum of the N-P operator. Let $\Omega_2 + Ra$ be such a translation, with unit vector a fixed and large parameter R. We join the boundary of Ω_1 to that of $\Omega_2 + Ra$ by a smooth curve γ and consider a tubular neighborhood U of γ whose width η will be chosen sufficiently small. The result of these operations, plus a local smoothing of the intersection of the boundaries of the these sets, is the domain

$$\Omega \approx \Omega_1 \cup U \cup (\Omega_2 + Ra).$$

We work first on the Lebesgue space $L^2(\partial\Omega)$, and consider there the N-P operator K associated to Ω. Let $\Gamma_1(\eta) = \partial\Omega_1 \setminus \overline{U}$ be the part of the boundary of Ω_1 which does not intersect the (smooth) joint with the tubular neighborhood U. The multiplication by the characteristic function of the set $\Gamma_1(\eta)$ defines a self-adjoint projector, denoted $P_1(\eta)$, acting on $L^2(\partial\Omega)$.

On the other hand, the boundary $\partial\Omega_1$ carries a Lebesgue space, the N-P operator K_1 and the same cut-off projector by the characteristic function of $\Gamma_1(\eta)$, still denoted $P_1(\eta)$. Moreover, the isometric identification

$$\|P_1(\eta) f\|_{\partial\Omega} = \|P_1(\eta) f\|_{\partial\Omega_1},$$

holds. Note that the projectors $P_1(\eta)$ converge strongly to the identity of $L^2(\partial\Omega_1)$ when η converges to zero. Since the operator K_1 is compact, the following norm convergence

$$\lim_{\eta \to 0} \|K_1 - P_1(\eta) K_1 P_1(\eta)\| = 0,$$

is true.

As a conclusion of these computations, and the general approximation principle stated at the beginning of the proof, we find that for η sufficiently small the spectrum of $P_1(\eta) K_1 P_1(\eta)$ approaches within distance ϵ the given finite part F_1 of the spectrum of K_1. Similarly, the projection $P_2(\eta)$ does the same service on the boundary of $\Omega_2 + Ra$.

Thus, the two "corners" $P_1(\eta)KP_1(\eta)$ and $P_2(\eta)KP_2(\eta)$ of the N-P operator of Ω have spectra in an ϵ neighborhood of the given set $F_1 \cup F_2$.

Next we use Poincaré's theorem, asserting that the spectrum of K can equally be computed via the energy form $\langle SK^*f, f\rangle = \langle KSf, f\rangle$, and the freedom to choose the translation parameter R large. Indeed,

$$\lim_{R\to\infty} \langle SP_1(\eta)f, P_2(\eta)\rangle = 0$$

for every pair of functions $f, g \in L^2(\partial\Omega)$. Likewise, using the energy norm interpretation of the boundary scalar product $\langle Sf, f\rangle$ we infer

$$\lim_{\eta\to 0}\langle S(P_1(\eta) + P_2(\eta))f, f\rangle = \|f\|^2.$$

Since the operator K is compact, we deduce that the differences

$$KS - (P_1(\eta)KSP_1(\eta) \oplus P_2(\eta)KSP_2(\eta))$$

and

$$[K - (P_1(\eta)KP_1(\eta) \oplus P_2(\eta)KP_2(\eta))]S$$

tend to zero uniformly, as soon as R becomes large and η small.

Thus, the spectrum of K can be approximated in the specified sense by the spectrum of $P_1(\eta)KP_1(\eta) \oplus P_2(\eta)KP_2(\eta)$ and this completes the proof of the theorem. □

Knowing that there are domains with negative spectrum as close to -1 as we desire, and the example of the unit ball (see Section 8.2), the preceding theorem shows that there are domains in $\mathbf{R}^d$ with at least as many finite negative and finite positive eigenvalues as one desires.

8. Examples

8.1. **The equilibrium distribution.** The notations are those adapted in the preliminaries. Let $(1, h) \in \mathfrak{H}$ be a single layer potential of the equilibrium distribution $\rho \in W^{-1/2}(\Gamma)$. Then by taking boundary values along Γ we find:

$$S^i_\rho = S^e_\rho = \mathbf{1}, \quad 0 = 2\partial_n S^i_\rho = \rho - K^*\rho.$$

Therefore

$$K\mathbf{1} = KS\rho = SK^*\rho = S\rho = \mathbf{1}.$$

If another function $f \in W^{1/2}(\Gamma)$ satisfies $Kf = f$, then there exists $\xi \in W^{-1/2}(\Gamma)$ such that $S\xi = f$ and by reversing the above identities we find $K^*\xi = \xi$, that is S_ξ produces zero energy inside Ω, hence it is a constant function. But this will imply that ξ is a scalar multiple of the equilibrium distribution.

Thus, for any closed smooth surface Γ, $\dim\ker(K - I) = 1$. On the other hand, always $\ker(K + I) = 0$.

Indeed, assume that $K\xi + \xi = 0$. That means $K^*S\xi + S\xi = 0$, that is $\partial_n S^e_\xi = 0$, which means that the field ∇S_ξ has zero energy on Ω_e. Thus $S^e_\xi = 0$ which implies $\xi = 0$.

8.2. **The ball in $\mathbf{R}^d$.** The complete solution of Poincaré's variational problem for the unit ball in $\mathbf{R}^3$ was given by Poincaré [34] . As this example is very important for our purposes let us briefly present it. We confine our attention to dimension $d = 3$, the cases $d > 3$ being analogous (there are however some anomalous aspects for $d = 2$).

In this section B denotes the unit ball in $\mathbf{R}^3$ and Γ its boundary. Let k be a non-negative integer, and denote by H_k the set of of homogeneous polynomials of degree n which satisfy the Laplace equation (augmented by 0, to make it a vector space). The dimension of H_k is $2k+1$, cf. [15]. For each F in H_k we can write $F(x) = r^k f(y)$ where $r = |x| := [\sum (x_i)^2]^{1/2}$, $y := x/r$ is a point of Γ, and f is a function on Γ, a so-called spherical harmonic of order k. Since $F(x)/r^{k+1}$ is harmonic in B_e (the exterior of Γ), the pair $u_i := r^k f(y)$ and $u_e := r^{-k-1} f(y)$ fit together continuously across Γ to form the (single layer) potential of a charge g on Γ. If $n = n_y$ denotes the outer normal to Γ at y, we have

$$\frac{\partial u_i}{\partial n} = kf(y), \quad \frac{\partial u_e}{\partial n} = -(k+1)f(y), \quad \text{along } \Gamma.$$

Thus $g(y) = (2k+1)f(y)$. For the field due to g, the part in B has energy equal to the integral of $(\partial_n u_i)g$ over Γ, that is $k(2k+1)\int_\Gamma f^2 d\sigma$, while the part in B_e has energy $(k+1)(2k+1)\int_\Gamma f^2 d\sigma$. Each of the pairs (u_i, u_e) so obtained is an extremal for the Poincaré problem insofar as the normal derivatives are proportional on Γ. The associated Neumann - Poincaré eigenvalue equals the ratio (with notations as earlier) $(J' - J)/(J' + J) = 1/(2k+1)$. To summarize:

The eigenvalues are the set $\{1, 1/3, 1/5, ...\}$ *, and to the eigenvalue* $1/(2k+1)$ *belongs an eigenspace of dimension* $2k+1$ *consisting of all spherical harmonics of order* k*.*

Since these eigenfunctions already span $L^2(\Gamma)$ there can be no other eigenvalues. In particular the spectral point 0 of the Neumann - Poincaré operator is not an eigenvalue , i.e. the Neumann - Poincaré' integral operator is injective in this case. Also, this can be seen at once since $K = \frac{1}{2}S$, see [7].

Observe also the remarkable consequence that for any charge of finite energy on Γ, the ratio of the energy of its field outside Γ to that of its field inside Γ exceeds $1/2$, and the value $1/2$ here is the largest possible.

The identity $K = \frac{1}{2}S$ also shows that the operator K does not satisfy the strongest S-symmetrization condition, namely that of possessing a continuous kernel (i.e. $K = LS$ with L compact), in the terminology of Krein [20].

The explicit spectral picture above implies that the N-P operator of the ball in $\mathbf{R}^d$, $d \geq 3$, is not trace-class. It would be interesting to decide whether there are domains with trace-class N-P operator.

For the disk in $\mathbf{R}^2$ a degeneracy occurs: For each $k > 0$, the space of "spherical harmonics" of order k is 2 - dimensional, spanned by $\sin(kt)$ and

$\cos(kt)$, where t is an angle variable along the unit circle, and for each f in the codimension one span of these, the field it engenders has equal energies inside and outside the unit circle. Here the Neumann - Poincaré operator has rank one. It is known [39] that the disk is the only planar domain for which the N - P operator has finite rank. It is not known whether there are such domains in higher dimensions.

Another characteristic property of the ball is discussed below.

Theorem 8.1. *The following is true: for a ball in* $\mathbf{R}^d$ *the N - P kernel is symmetric, and balls are the only domains with this property.*

Proof. For the proof let us confine attention to (smoothly bounded) domains in $\mathbf{R}^3$. The argument is nearly identical in all dimensions. Apart from a constant of normalization the kernel in question is

$$K(x,y) = \frac{(x-y)\cdot n(y)}{\|x-y\|^3}, \quad x,y \in \Gamma,$$

where C denotes the boundary of the domain Ω under consideration, and $n(y)$ denotes the unit outer normal to Γ at y. The symmetry of K means

$$(x-y)\cdot n(y) = (y-x)\cdot n(x), \quad \text{for all } x,y \in \Gamma,$$

i.e.

(*)*For any two distinct points* x, y *of* Γ *the vector sum of the unit outer normals to* Γ *at* x *and* y *is perpendicular to the chord joining* x *and* y*.*

It is easy to check that spheres enjoy this property, so let us turn to the converse. Assume Γ has property (*). We shall show it is a sphere. First note the following two immediate consequences of (*):

(i) If the normal to Γ at some point has other intersections with Γ, it coincides with the normal at each of those points.

(ii) If the normals to Γ at two distinct points x, y intersect at a point z, the distances of z from x and y are equal. (Indeed , it follows from (*) that the triangle formed by x, y, z is isosceles, having equal angles at x and y.)

We now conclude the proof that Γ is a sphere. Let x be any point inside Γ, and y a point of Γ at minimal distance from x. The line L joining y to x is orthogonal to Γ at y, and meets Γ in another point z distinct from y , where again it is orthogonal to Γ, by (i). Let now w denote the midpoint of the chord joining y and z. We claim Γ is a sphere centered at w, with radius r equal to half the length of the chord joining y and z. Indeed, suppose there is a point of Γ at distance from w unequal to r, say greater than r. Then a point u exists on Γ at maximal distance s from w, where $s > r$. A moment's thought implies that u cannot be collinear with w and z, and the line joining w to u meets Γ orthogonally. Hence (ii) applies, and yields that w is equidistant from z and u, which is a contradiction. This concludes the proof. □

Remarks. It may be of some interest to try to characterize those domains whose N - P kernel satisfies various weaker symmetry assumptions,

such as:

a) K is symmetric "modulo rank 1" , i.e. it is a symmetric kernel plus a "perturbation" of the form $a(x)b(y)$.

b) The iterated (or, m times iterated) kernel associated to K is symmetric.

Yet another aspect of symmetrization of the kernel (in two dimensions) based on change of independent variable, was discussed in an interesting paper [10] by D. Gaier.

8.3. **The ellipse.** An analysis of the single and double layer potential operators on an ellipse goes back to Neumann [31]. The computations, also reproduced in the book by Plemelj [33], start with the elliptical coordinates:

$$x(t) = E\cosh\rho\ \cos t, \qquad y(t) = E\sinh\rho\ \sin t,$$

where ρ and E are positive parameters. The half axes of the ellipse are

$$a = E\cosh\rho, \quad b = E\sinh\rho.$$

Let

$$q = e^{-2\rho} = \frac{a-b}{a+b} < 1$$

be the eccentricity of the ellipse. For two points $z(t) = (x(t), y(t)),\ z(s) = (x(s), y(s))$ on the ellipse one computes by elementary means

$$\arg(z(t) - z(s)) = \arctan\frac{y(t)-y(s)}{x(t)-x(s)} = \frac{s+t+\pi}{2} + \sum_{k=1}^{\infty}\frac{q^k}{k}\sin k(s+t),$$

and

$$\log|z(t) - z(s)| = \log|(a+b)\sin\frac{s-t}{2}| - \sum_{k=1}^{\infty}\frac{q^k}{k}\sin k(s+t).$$

By differentiation one identifies the kernel $K(s,t)$ of the Neumann-Poincaré operator:

$$K(s,t) = 1 + 2\sum_{k=1}^{\infty} q^k\cos k(s+t).$$

From here, the spectrum can be identified by standard Fourier methods.

Proposition 8.2. *The spectrum of the Neumann-Poincaré operator on an ellipse of eccentricity $q < 1$ is $\{\pm q^k;\ k \geq 1\} \cup \{0, 1\}$.*

The same conclusion was reached by Bergman and Schiffer [3] via the operator T_Ω, the associated L-kernel, and a conformal mapping on the complement of the ellipse.

8.4. **Lemniscates.** Let P denote a polynomial in one complex variable with complex coefficients (of degree at least 1) and M a sufficiently large positive number that the curve $\Gamma = \{z;\ |z| = M\}$ encloses all the roots of P. We'll prove:

Theorem 8.3. *The Neumann - Poincaré operator for the domain enclosed by* Γ *has an infinite dimensional kernel.*

Proof. It is no loss of generality to suppose $M = 1$. Write $P = u+iv$ where u and v are real harmonic polynomials. Then, on Γ we have $u+iv = 1/(u-iv)$. Therefore the pair of harmonic functions

$$u \text{ on } \Omega, \qquad u/(u^2+v^2) \text{ on } \Omega_e,$$

where Ω, Ω_e denote respectively the interior and exterior domains determined by Γ, have matching boundary values on Γ. Together they constitute the potential of a charge supported on Γ and having finite energy. As we have seen, this charge is in the kernel of the N - P operator if and only if the normal derivatives of these functions on Γ (w.r.t. say the outer normal n) are everywhere negatives of each other, that is

$$\partial_n u + (u^2+v^2)\partial_n u - u(2u\partial_n u + 2v\partial_n v) = 0 \ \text{ on } \Gamma, \tag{39}$$

or, simplifying

$$(1-u^2)\partial_n u - uv\partial_n v = 0 \ \text{ on } \Gamma.$$

Substituting $1-u^2 = v^2$ and cancelling v, this becomes

$$v\partial_n u = u\partial_n v \quad \text{on } \Gamma.$$

By virtue of the Cauchy - Riemann equations, $\partial_n u = \partial_\tau v$ and $\partial_\tau u = -\partial_n v$, where τ denotes the unit tangent vector to Γ. Thus, the last equation is equivalent to $u\partial_\tau u + v\partial_\tau v = 0$ along Γ. But, this is true, it is just the result of differentiating $u^2+v^2 = 1$ along Γ in the tangential direction. Since all the steps are reversible, (39) is proved and the charge defined by the above potential is indeed in the kernel. Applying the identical procedure, but starting in turn with the polynomials $P^2, P^3, P^4, ...$ we get infinitely many elements in the kernel. These are linearly independent, since their potentials all have different rates of decay at ∞. The theorem is proved. □

Remark. The multiplication trick at the end to get the infinite dimensionality seems to have no counterpart in more than 2 dimensions. Although the first part of the proof also was heavily dependent on two dimensional features (harmonic conjugates and Cauchy - Riemann) it does not seem beyond credibility that an analogous example could be found in 3 or more dimensions, i.e. an example where the kernel is nontrivial.

More precisely, calculations analogous to those above lead to the conclusion that a sufficient condition for existence of a domain in $\mathbf{R}^3$ with non-injective N - P operator is the affirmative resolution of the following

Hypothesis. There are three real polynomials p, q, r of 3 variables satisfying the following conditions:

a) p is harmonic;

b) p, q, r have no common zero on $\mathbf{R}^3$;

c) $s := p^2 + q^2 + r^2$ tends to ∞ at ∞;

d) p/s is harmonic;

e) Denoting by Ω a nonempty component of the set $\{s < 1\}$ we have along the surface $\partial\Omega$ the identity

$$\frac{\partial_n p}{p} = \frac{\partial_n q}{q} = \frac{\partial_n r}{r}.$$

One could presumably write a computer program to search for such a triple among low degree polynomials.

Note that already for the disk the closure of the single layer potentials which belong to $\ker K$ in the energy norm coincides with $W^{1/2}$ modulo constants, i.e. with the subspace in $W^{1/2}$ consisting of all functions on the circle with the mean value zero. This explains the painstaking caution one must obey in the statement of Theorem 4.2: unlike for eigenfunctions corresponding to nonzero eigenvalues one cannot really expect much additional regularity, e.g. membership in $W^{1/2}$, for mass distributions χ for which $S_\chi \in \ker K$.

Furthermore, for general lemniscates we still do not know whether the closure of $\Re P^n, \Im P^n,\ n = 1, 2, ...$ (cf. the notation in the proof of Theorem 8.3) covers all of $\ker K$, i.e. whether the preimages of these functions with respect to the operator S are dense in the space of all distributions $\chi \in W^{-1/2}$ such that $S_\chi \in \ker K$ (cf. Theorem 4.2).

Even less is known in higher dimensions. As we saw (Section 8.1) in the ball $\ker K = 0$. We do not know any particular example of a bounded domain in $\mathbf{R}^d$ with a nontrivial $\ker K$, yet we strongly suspect that there are such domains. For unbounded domains the situation is completely different, e.g., for the half-space K is simply a trivial zero operator, so it is kernel is all of L^2.

Notations:

Ω is a bounded domain of $\mathbf{R}^d$;

$\Omega_i = \Omega$, $\Omega_e = \mathbf{R}^d \setminus \overline{\Omega}$;

$\Gamma = \partial\Omega$ the boundary of Ω, assumed to be smooth of class at least C^2;

$d\sigma(s)$ the $d-1$ volume measure on Γ;

n or $n_s = n(s)$ the outer (unit) normal to a point $s \in \Gamma$; $\partial_n = \frac{\partial}{\partial n}$;

$E(x)$ the fundamental solution to the Laplace operator Δ: $-\Delta E = \delta$;

$(Sf)(x) = \int_\Gamma E(x-y)f(y)d\sigma(y)$ the single layer potential operator on $L^2(\Gamma)$;

$S_\rho(x) = \rho_y(E(x,y)), x \in \mathbf{R}^d \setminus \Gamma$, the single layer potential of a distribution $\rho \in \mathcal{D}'(\Gamma)$;

$(Kf)(x) = -2\int_\Gamma \partial_{n(y)}E(x-y)f(y)d\sigma(y)$ the double layer potential operator on $L^2(\Gamma)$;

$(D_f)(x) = \rho_y(\partial_{n_y}E(x,y)), x \in \mathbf{R}^d \setminus \Gamma$, the double layer potential of a distribution $\rho \in \mathcal{D}'(\Gamma)$;

$W_F^s(U) = W_F^{s,2}(U)$ the Sobolev space of order $(s,2)$, on the open set U, with supports in F;

$\mathfrak{H}$ the Hilbert space of pairs $h = (h_i, h_e)$of harmonic functions in Ω_i, Ω_e with norm $\|(h_i,h_e)\|_{\mathfrak{H}}^2 = \int_{\Omega_i}|\nabla h_i|^2dx + \int_{\Omega_e}|\nabla h_e|^2dx$;

$J[h] = J_i[h] = \int_{\Omega_i}|\nabla h_i|^2dx$, $J'[h] = J_e[h] = \int_{\Omega_e}|\nabla h_e|^2dx$, the inner, respectively outer, energies of the potential h;

$\mathfrak{S} \oplus \mathfrak{D} = \mathfrak{H}$ the orthogonal decomposition into the subspaces of single (respectively double) layer potentials .

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