Grön: $\vec{r}\,'(t) = (\sin t,\ \sin t^2 + 2t^2\cos t,\ 2t)$

1. $= (0,0,0)$ om $t = 0$

singulär punkt

$\vec{r}(0) = (1-1,\ 0\cdot\sin 0^2,\ 0^2+1)$

$= (0,0,1)$

2. $\operatorname{grad}(xy^2 + zy^3 - z^2) =$

$(y^2,\ 2xy + 3zy^2,\ y^3 - 2z) \Rightarrow$

$= (0,0,0)$ om $(x,0,0)$

$x0^2 + 0 \cdot 0^3 - 0^2 = 0 \neq 3$

SVAR: reguljär

3. $\lim_{x \to 0} \frac{x^3}{x^2} = 0$, $\lim_{y \to 0} \frac{y^2}{y^2} = 1$

existerar ej.

rosa: $\vec{r}(t) = (3t - t^3,\ \cos(2\pi t),\ t^2 + 2t)$

$\vec{r}'(t) = (3 - 3t^2,\ -2\pi \sin(2\pi t),\ 2(t+1))$

singulär när $t = -1$

$(-3+1,\ \cos(-2\pi),\ 1-2)$

$= (-2,\ 1,\ -1)$

2. $x^2 - 3xy - z^3 = 2$

$\text{grad}(x^2 - 3xy - z^3) = (2x - 3y,\ -3x,\ -3z^2)$

$= (0,0,0)$ bara om $(x,y,z) = (0,0,0)$ men

$0^2 - 3 \cdot 0 \cdot 0 - 0^3 = 0 \neq 2$

3. $\lim_{x\to 0} \frac{x^2}{x^2} = 1$, $\lim_{y\to 0} \frac{y^2}{y^2} = 1$

$x=y$ $\lim_{x\to 0} \frac{x^2-3x^2+x^2}{x^2+x^2} = -\frac{1}{2}$

gränsvärdet existerar ej.

Svar: 1. $\vec{r}(t) = (\cos t^2, t\sin t, t^2)$

$\vec{r}'(t) = (-2t\sin t^2, \sin t + t\cos t, 2t)$

$= (0,0,0) \Rightarrow t = 0$

singulär punkt $(1,0,0)$

2. $x^3 - 2y^2 + 4z^2 = 1$

$\text{grad}\, F = (3x^2, -4y, 8z)$

$= (0,0,0)$ bara för $(x,y,z) = (0,0,0)$

men $0^3 - 2 \cdot 0^2 + 4 \cdot 0^2 = 0 \neq 1$

3. $\lim_{x \to 0} \frac{x^2}{x^2} = 1$, $\lim_{y \to 0} \frac{y^2}{y^2} = 1$

$x = y$ $\lim_{x \to 0} \frac{x^2 + x^2 + x^2}{x^2 + x^2} = \frac{3}{2}$

Existerar ej!

Högre derivator

Beteckning: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Ex: Låt $f(x,y) = x^2 + 3xy + 2y^2$.

Vi har att

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x + 4y\right) = 3$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(2x+3y\right) = 3$$

Vi ser att $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ i det här fallet men det gäller också allmänt.

Sats: Om de blandade andra-derivatorna $\frac{\partial^2 f}{\partial x \partial y}$ och $\frac{\partial^2 f}{\partial y \partial x}$ är kontinuerliga i en omgivning av en punkt så är de lika i omgivningen.

Man måste vara lite försiktig:

Ex 4.10: Låt $f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

(Så $f(x,y) = xy\,g(x,y)$ där $\lim_{x\to 0} g(x,0) \neq \lim_{y \to 0} g(0,y)$)

Vi får

$(x,y) \neq (0,0)$ $\quad \dfrac{\partial f}{\partial x}(x,y) = \dfrac{(3x^2y - y^3)(x^2+y^2) - 2x(x^3y - xy^3)}{(x^2+y^2)^2}$

$= \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$

Så $\displaystyle \frac{\partial f}{\partial x}(0,y) = -\frac{y^5}{y^4} = -y$

och $\displaystyle \frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = 0$

$(x,y)\neq(0,0)$: $\displaystyle \frac{\partial f}{\partial y}(x,y) = \frac{(x^3-3xy^2)(x^2+y^2)-2y(x^3y-xy^3)}{(x^2+y^2)^2}$

$$\frac{\partial f}{\partial y}(x,0) = \frac{x^5}{x^4} = x$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = 0$$

Alltså

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{x \to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} =$$

$$= \lim_{x \to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{y \to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} =$$

$$= \lim_{y \to 0} \frac{-y - 0}{y} = -1$$

Så $\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$!