Koordinattransformationer och differentialekvationer

Ex: Om f är en funktion av en variabel och vi sätter

$$z(x,y) = f(x+2y)$$ ser vi att

$$\frac{\partial z(x,y)}{\partial x} = f'(x+2y)\cdot 1$$

$$\frac{\partial z(x,y)}{\partial y} = f'(x+2y)\cdot 2$$

Så $2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$, dvs

$z = f(x+2y)$ är en lösning till differentialekvationen. Vi vill nu visa

att $z(x,y) = f(x+2y)$ ger den allmänna lösningen. Vi inför

koordinaterna $T: \begin{cases} u = x+2y \\ v = y \end{cases}$

dvs $T: \mathbb{R}^2 \to \mathbb{R}^2$. $\Leftrightarrow \begin{cases} x = u-2v \\ y = v \end{cases} : T^{-1}$

Låt $\tilde{z}(u(x,y), v(x,y)) = z(x,y)$.

Kedjeregeln ger så

$$\operatorname{grad} z(x,y) = \operatorname{grad} \tilde{z}(u,v)\, J_T$$

$$J_T = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Vi får alltså

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = \left(\frac{\partial \tilde{z}}{\partial u}, \frac{\partial \tilde{z}}{\partial v}\right)\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \left(\frac{\partial \tilde{z}}{\partial u}, 2\frac{\partial \tilde{z}}{\partial u} + \frac{\partial \tilde{z}}{\partial v}\right)$$

$$2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$ transformeras

till

$$0 = 2\cancel{\frac{\partial \tilde{z}}{\partial u}} - 2\cancel{\frac{\partial \tilde{z}}{\partial u}} - \frac{\partial \tilde{z}}{\partial v} \Leftrightarrow \frac{\partial \tilde{z}}{\partial v} = 0$$

Så $\tilde{z}(u,v)$ beror ej av v. Alltså

$$\tilde{z}(u,v) = f(u) \Rightarrow$$

$$z(x,y) = f(x+2y)$$

$$2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\left(\frac{\partial \tilde{z}}{\partial u}, \frac{\partial \tilde{z}}{\partial v}\right)\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \left(\frac{\partial \tilde{z}}{\partial u}, \frac{\partial \tilde{z}}{\partial v}\right)\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= -\frac{\partial \tilde{z}}{\partial v}$$

Det viktiga när man byter koordinater är att transformationen är inverterbar.

$$T: \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases} \qquad T^{-1}: \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$T^{-1} \circ T(x,y) = T^{-1}(T(x,y)) = T^{-1}(u,v) = (x,y)$$

så

$$J_{T^{-1}\circ T}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

men kedjeregeln ger

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = J_{T^{-1}\circ T}(x,y) = J_{T^{-1}}(u,v)\, J_T(x,y)$$

så

$$J_{T^{-1}}(u,v) = J_T(x,y)^{-1}$$

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}^{-1}$$

Ex: I vårt tidigare exempel skulle vi alltså få

$$\left(\frac{\partial\tilde{z}}{\partial u}, \frac{\partial\tilde{z}}{\partial v}\right) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^{-1} =$$

$$= \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \left(\frac{\partial z}{\partial x}, -2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right)$$

Så $\frac{\partial\tilde{z}}{\partial v}$ svarar mot $-2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$.

Ex 4.17) Använd transformationen

$$T: \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$ för att lösa

differentialekvationen

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$

Lösn: Kedjeregeln ger

$$\left(\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \varphi}\right) = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} =$$

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$$= \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$\Rightarrow$

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \varphi}\right) \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}^{-1} =$$

$$= \left(\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \varphi}\right) \frac{1}{r} \begin{pmatrix} r\cos\varphi & r\sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

$\Leftrightarrow$

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = \left(\frac{\partial z}{\partial r}\cos\varphi - \frac{1}{r}\frac{\partial z}{\partial \varphi}\sin\varphi, \frac{\partial z}{\partial r}\sin\varphi + \frac{1}{r}\frac{\partial z}{\partial \varphi}\cos\varphi\right)$$

Alltså

(∗) $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$ svarar mot

$$0 = r\sin\varphi\left(\frac{\partial z}{\partial r}\cos\varphi - \frac{1}{r}\frac{\partial z}{\partial \varphi}\sin\varphi\right) - r\cos\varphi\left(\frac{\partial z}{\partial r}\sin\varphi + \frac{1}{r}\frac{\partial z}{\partial \varphi}\cos\varphi\right)$$

$$= (\underbrace{r\sin\varphi\cos\varphi - r\sin\varphi\cos\varphi}_{=0})\frac{\partial z}{\partial r} -$$

$$-(\underbrace{\sin^2\varphi + \cos^2\varphi}_{=1})\frac{\partial z}{\partial \varphi} = -\frac{\partial z}{\partial \varphi}$$

Så en lösning till (∗) beror bara av r, dvs

$$z(x,y) = f(\underbrace{\sqrt{x^2+y^2}}_{=r}) = g(x^2+y^2)$$

Ska man räkna högre derivator är det lämpligt att använda operatornotation: (i exemplet)

$$\frac{\partial}{\partial x} = \cos\varphi \frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi \frac{\partial}{\partial \varphi}$$

$$\frac{\partial}{\partial y} = \sin\varphi \frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi \frac{\partial}{\partial \varphi}$$

Om vi exempelvis vill transformera

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$ så blir det

$$\Big(\cos\varphi \overbrace{\frac{\partial}{\partial r}}^{\frac{\partial}{\partial x}} - \frac{1}{r}\sin\varphi\frac{\partial}{\partial\varphi}\Big)\Big(\sin\varphi \overbrace{\frac{\partial z}{\partial r} + \frac{1}{r}\cos\varphi\frac{\partial z}{\partial\varphi}}^{\frac{\partial z}{\partial y}}\Big) =$$

$$= \Big(\cos\varphi\sin\varphi\frac{\partial^2 z}{\partial r^2} + \cos\varphi\frac{\partial}{\partial r}\Big(\frac{1}{r}\cos\varphi\frac{\partial z}{\partial\varphi}\Big) -$$

$$- \frac{1}{r}\sin\varphi\frac{\partial}{\partial\varphi}\Big(\sin\varphi\frac{\partial z}{\partial r}\Big) - \frac{1}{r^2}\sin\varphi\frac{\partial}{\partial\varphi}\Big(\cos\varphi\frac{\partial z}{\partial\varphi}\Big)\Big)$$

$$= \Big(\cos\varphi\sin\varphi\frac{\partial^2 z}{\partial r^2} + \cos^2\varphi\Big(-\frac{1}{r^2}\frac{\partial z}{\partial\varphi} + \frac{1}{r}\frac{\partial^2 z}{\partial r\partial\varphi}\Big) -$$

$$- \frac{1}{r}\sin\varphi\Big(\cos\varphi\frac{\partial z}{\partial r} + \sin\varphi\frac{\partial^2 z}{\partial\varphi\partial r}\Big) - \frac{1}{r^2}\sin\varphi\Big(-\sin\varphi\frac{\partial z}{\partial\varphi} + \cos\varphi\frac{\partial^2 z}{\partial\varphi^2}\Big)$$

$$= \cos\varphi\sin\varphi\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}(\cos^2\varphi - \sin^2\varphi)\frac{\partial^2 z}{\partial r\partial\varphi} - \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial^2 z}{\partial\varphi^2} -$$

$$- \frac{1}{r^2}(\cos^2\varphi - \sin^2\varphi)\frac{\partial z}{\partial\varphi} - \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial z}{\partial r}.$$

Ex: $\left(x\frac{d}{dx}\right)\left(x\frac{df}{dx}\right) = x\underbrace{\frac{d}{dx}\left(x\frac{df}{dx}\right)} =$ $g(x)=\frac{df}{dx}$

$= x\frac{df}{dx} + x^2\frac{d^2f}{dx^2}$

$\frac{d}{dx}(xg)$

$= \frac{dx}{dx}g + x\frac{dg}{dx}$

$= g + x\frac{dg}{dx}$

Derivation av vissa integraler

Vi börjar med en sats

Sats: Om $f(x,y)$ och $\frac{\partial f}{\partial x}(x,y)$ är kontinuerliga i området $a \leq x \leq b$, $c \leq y \leq d$ så är

$$\frac{d}{dx}\left(\int_c^d f(x,y)\,dy\right) = \int_c^d \frac{\partial f(x,y)}{\partial x}\,dy$$

Ex: $$\frac{d}{dx}\left(\int_0^1 \frac{\sin(xy)}{y}\,dy\right) = \int_0^1 \frac{\partial}{\partial x}\left(\frac{\sin xy}{y}\right) dy =$$

$$= \int_0^1 \frac{\cancel{x}\cos(xy)}{\cancel{x}}\,dy = \int_0^1 \cos(xy)\,dy = \frac{\sin x}{x}$$

Sats: Om $\alpha(x)$, $\beta(x)$ är deriverbara funktioner med värden i intervallet $c \leq y \leq d$ och $f(x,y)$ och $\frac{\partial f}{\partial x}(x,y)$ är kontinuerliga då $a \leq x \leq b$, $c \leq y \leq d$, så är

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$$\frac{d}{dx}\left(\int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\right) =$$

$$= \int_{\alpha(x)}^{\beta(x)} \frac{\partial f}{\partial x}(x,y)\,dy + f(x,\beta(x))\,\beta'(x) - f(x,\alpha(x))\,\alpha'(x)$$

Ex 4.22)
$$\frac{d}{dx}\left(\int_{x}^{x^2} \frac{e^{xy}}{y}\,dy\right) = \int_{x}^{x^2} \frac{\partial e^{xy}}{\partial x}\frac{dy}{y} + \frac{e^{x^3}}{x^2}2x -$$

$$- \frac{e^{x^2}}{x}\cdot 1 = \int_{x}^{x^2} e^{xy}\,dy + \frac{2e^{x^3}}{x} - \frac{e^{x^2}}{x}$$

$$= \left[\frac{e^{xy}}{x}\right]_{x}^{x^2} + 2\frac{e^{x^3}}{x} - \frac{e^{x^2}}{x} = 3\frac{e^{x^3}}{x} - 2\frac{e^{x^2}}{x}$$