gr

1. $f(x,y) = x^3 + 2y^2$

$(1,2) \longrightarrow (3,2)$

enhetsvektor $\dfrac{(3,2)-(1,2)}{|(3,2)-(1,2)|} = \dfrac{(2,0)}{2} = (1,0)$

$\text{grad} f(1,2) = (3 \cdot 1^2,\ 4 \cdot 2) = (3,8)$

$\text{grad} f(1,2) \cdot (1,0) = 3$

rosa

1. $f(x,y) = x^2 + xy^2$

$(0,2) - (1,1) = (-1,1)$

enhetsvektor $\dfrac{(-1,1)}{\sqrt{2}}$

$\operatorname{grad} f(1,1) = (2\cdot 1 + 1^2, 2\cdot 1 \cdot 1) = (3,2)$

$\operatorname{grad} f(1,1) \cdot \dfrac{(-1,1)}{\sqrt{2}} = \dfrac{-3+2}{\sqrt{2}} = -\dfrac{1}{\sqrt{2}}$

Grad

$$x^2y + xy^2$$

$$(1,3) \to (2,2)$$

$$(2,2) - (1,3) = (1,-1)$$

enhetsvektor $\dfrac{(1,-1)}{\sqrt{2}}$

$$\operatorname{grad} f(1,3) \cdot \frac{(1,-1)}{\sqrt{2}} = (2 \cdot 1 \cdot 3 + 3^2,\ 1^2 + 2 \cdot 1 \cdot 3) \cdot \frac{(1,-1)}{\sqrt{2}}$$

$$= \frac{(15,7) \cdot (1,-1)}{\sqrt{2}} = \frac{8}{\sqrt{2}}$$

Grå

2. $f(x,y) = (\sin(xy),\ x-y)$

$$\left.\begin{vmatrix} y\cos(xy) & x\cos(xy) \\ 1 & -1 \end{vmatrix}\right|_{(x,y)=(1,1)} =$$

$$\begin{vmatrix} \cos 1 & \cos 1 \\ 1 & -1 \end{vmatrix} = -2\cos 1 \neq 0$$

Rasa

2. $f(x,y) = \left(y^2 \sin(x),\ \cos(x+y)\right)$

$$\left. \begin{vmatrix} y^2\cos x & 2y\sin x \\ -\sin(x+y) & -\sin(x+y) \end{vmatrix} \right|_{(x,y)=(2,-1)} =$$

$$= \begin{vmatrix} \cos 2 & -2\sin 2 \\ -\sin 1 & -\sin 1 \end{vmatrix} = -\sin 1 \left(\cos 2 + 2\sin 2\right)$$

$$\neq 0$$

Gul

2. $f(x,y) = (e^{x+y},\ x+y^2)$

$$\left|\begin{matrix} e^{x+y} & e^{x+y} \\ 1 & 2y \end{matrix}\right|_{(1,2)} = \left|\begin{matrix} e^3 & e^3 \\ 1 & 4 \end{matrix}\right| = 3e^3 \neq 0$$

Svi

2. $F(x,y,z) = x^2z^2 - xyz$

$$H_{(2,1)}(z) = 4z^2 - 2z - 6$$

$$H'_{(2,1)}(-1) = 8\cdot(-1) - 2 = \underline{\underline{-10}} \neq 0$$

$$0 = 2xz^2 + x^2 2z z'_x - yz - xyz'_x$$

$$z'_x(2,1) = \left.\frac{yz - 2xz^2}{\underset{-8}{x^2 2z} \underset{-2=-10}{-xy}}\right|_{(2,1,-1)} = \frac{1}{2}$$

rosa

7. $xz^3 - yz^2 - 4 = 0$

$$H_{(1,1)}(z) = z^3 - z^2 - 4$$

$$H'_{(1,1)}(2) = 3\cdot 2^2 - 2\cdot 2 = 8$$

$$z'_x = -\frac{z^3}{3xz^2 - 3yz} = -\frac{8}{12-4} = -1$$

$$z'_y = \frac{1}{2}$$

Gul

3. $xyz^3 + z - 3$

$$H_{(1,2)}(z) = 2z^3 + z - 3$$

$$H'_{(1,2)}(1) = 6 \cdot 1^2 + 1 = 7 \neq 0$$

$$z'_x = -\frac{yz^3}{xy3z^2+1} = -\frac{2}{7}$$

$$z'_y = -\frac{1}{7}$$

Taylors formel i en variabel

Taylors formel ger en approximation av en funktion med hjälp av polynom

$$f(x+h) = P_n(h) + R_{n+1}(h)$$

$$P_n(h) = f(x) + \ldots + \frac{f^{(n)}(x)}{n!} h^n$$

$$R_{n+1}(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

$h > 0$: $x < \xi < x+h$

$x+h < \xi < x$: $h < 0$

$$f(x) = p_n(0), \quad f'(x) = p_n'(0)$$

$$\ldots \quad f^{(n)}(x) = p_n^{(n)}(0)$$

Med andra beteckningar: $(\tilde{x} = x+h,\ a = x)$

$$f(\tilde{x}) = f(a) + f'(a)(\tilde{x}-a) + \ldots + \frac{f^{(n)}(a)}{n!}(\tilde{x}-a)^n + R_{n+1}(\tilde{x}-a)$$

$$O\left(|\tilde{x}-a|^{n+1}\right)$$

Några exempel:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!} + O(t^{n+1})$$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + O(t^{2n+2})$$

$$\cos t = 1 - \frac{t^2}{2!} + \ldots + (-1)^n \frac{t^{2n}}{(2n)!} + O(t^{2n+1})$$

MacLaurinutvecklingar, dvs

Taylorutveckling i $t=0$.

Ex: utveckla e^t kring $t=1$

$$e^t = e^{t-1}e = \left[s=t-1\right] = e \cdot e^s =$$

$$= e\left(1 + s + \frac{s^2}{2!} + \dots + \frac{s^n}{n!} + O\left(s^{n+1}\right)\right) =$$

$$= e + e(t-1) + e\frac{(t-1)^2}{2!} + \dots + e\frac{(t-1)^n}{n!} + O\left((t-1)^{n+1}\right)$$

Ex: utveckla $\sin t \cos t$ till ordning 5 kring origo:

$$\sin t\cos t = \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + O(t^7)\right)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + O(t^6)\right) =$$

$$= t - \frac{t^3}{2!} - \frac{t^3}{3!} + \frac{t^5}{4!} + \frac{t^5}{3!\,2} + \frac{t^5}{5!} + O(t^6) =$$

$$= t - \frac{2t^3}{3} + \frac{t^5}{2\cdot 3!}\left(\frac{1}{2} + \frac{1}{1} + \frac{1}{10}\right) + O(t^6)$$

$$= t - \frac{2t^3}{3} + \frac{2t^5}{15} + O(t^6)$$

$$f(t) = \sin t \cos t$$

$$f^{(5)}(0) = \frac{2\cdot 5!}{15} = 16$$