Determinanter och produkter

Ex: Låt $A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

och $C = A \cdot B = \begin{pmatrix} 2 & 10 \\ 1 & 10 \end{pmatrix}$.

Vi får $\det C = 2 \cdot 10 - 1 \cdot 10 = 10$

och $\det A = 2 \cdot 4 - 1 \cdot 3 = 5$, $\det B = 1 \cdot 2$.

Vi ser att $\det A \cdot \det B = \det C$.

Sats: Om A och B är två $n \times n$-matriser så gäller det att

$$\det(AB) = \det A \cdot \det B.$$

Eftersom multiplikation av reella tal är kommutativ får vi även

Kor: $\det AB = \det BA$

Bevis av satsen: Om $C = AB$ har vi att

$$\det C = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \, c_{1\sigma(1)} \cdots c_{n\sigma(n)},$$

men $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$ så vi får

$$\det C = \sum_{\sigma \in S_n} \operatorname{sgn}\sigma \left(\sum_{k_1=1}^{n} a_{1k_1} b_{k_1\sigma(1)} \right) \cdots \left(\sum_{k_n=1}^{n} a_{nk_n} b_{k_n\sigma(n)} \right)$$

$$\begin{pmatrix}\text{två lika}\\ k_i \text{ ger}\\ \text{noll}\end{pmatrix} = \sum_{\sigma\in S_n} \operatorname{sgn}\sigma \sum_{\tau\in S_n} a_{1\tau(1)}\cdots a_{n\tau(n)}\cdot$$

$$\cdot b_{\tau(1)\sigma(1)}\cdots b_{\tau(n)\sigma(n)} =$$

$$\begin{pmatrix}\eta=\sigma\tau^{-1}\\ \operatorname{sgn}\eta=\\ \operatorname{sgn}\sigma\operatorname{sgn}\tau\end{pmatrix} = \left(\sum_{\tau\in S_n}\operatorname{sgn}\tau\, a_{1\tau(1)}\cdots a_{n\tau(n)}\right)\cdot$$

$$\cdot\left(\sum_{\eta\in S_n}\operatorname{sgn}\eta\, b_{1\eta(1)}\cdots b_{n\eta(n)}\right)$$

$$= \det A\cdot\det B.$$

2×2-matriser:

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$$

$$= a_{11}a_{22}b_{11}b_{22} + a_{12}a_{21}b_{21}b_{12} -$$

$\tau, \sigma = id, \eta = id$ $\quad$ $\tau = (12), \sigma = id, \eta = (12)$

$$- a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{22}b_{11}$$

$\tau = id, \sigma = (12), \eta = (12)$ $\quad$ $\tau = (12), \sigma = (12)$ $\eta = id$

$$= (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$$

Övn: Gå igenom exemplet

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix},\ B = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix},\ C = \begin{pmatrix} 2 & 10 \\ 1 & 10 \end{pmatrix}.$$

Adjunkter

Ex: Determinanten $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}$ kan beräknas

genom att utveckla längs första raden

$$1 \cdot \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} = 5 - 2 + 3(-7) = -18$$

Om $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ så har vi

$$\det A = a_{11} \underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{C_{11}} - a_{12} \underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{C_{12}} + a_{13} \underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{C_{13}}$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \begin{pmatrix} C_{11} \\ -C_{12} \\ C_{13} \end{pmatrix}$$

$$\begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} c_{11} \\ -c_{12} \\ c_{13} \end{pmatrix} = 0$$

eftersom det svarar mot $\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$

Vi ser att

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} c_{11} \\ -c_{12} \\ c_{13} \end{pmatrix} = \begin{pmatrix} a_{11}c_{11} - a_{12}c_{12} + a_{13}c_{13} \\ a_{21}c_{11} - a_{22}c_{12} + a_{23}c_{13} \\ a_{31}c_{11} - a_{32}c_{12} + a_{33}c_{13} \end{pmatrix} = \begin{pmatrix} \det A \\ 0 \\ 0 \end{pmatrix}$$

och
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \underbrace{\begin{pmatrix} C_{11} & -C_{21} & C_{31} \\ -C_{12} & C_{22} & -C_{32} \\ C_{13} & -C_{23} & C_{33} \end{pmatrix}}_{\substack{\text{adjunkten} \\ \operatorname{adj} A}} = \begin{pmatrix} \det A & 0 & 0 \\ 0 & \det A & 0 \\ 0 & 0 & \det A \end{pmatrix}$$

Om $\det A \neq 0$ är matrisen inverterbar och

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A .$$

Ex: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\operatorname{adj} A = \begin{pmatrix} C_{11} & -C_{21} \\ -C_{12} & C_{22} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

så $A^{-1} = \dfrac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ $\quad \det A = \left| \begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 2 & 3 & 1 & 2 & 3 \\ 3 & 1 & 2 & 3 & 1 \end{array} \right|$

$= 6 + 6 + 6 - 27 - 1 - 8 = -18$

$$\operatorname{adj} A = \begin{pmatrix} C_{11} & -C_{21} & C_{31} \\ -C_{12} & C_{22} & -C_{32} \\ C_{13} & -C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \end{pmatrix} =$$

$$= \begin{pmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{pmatrix} \Rightarrow A^{-1} = -\frac{1}{18}\begin{pmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{pmatrix}$$

Test: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{pmatrix} =$

$$= \begin{pmatrix} 1\cdot 5 - 2\cdot 1 - 3\cdot 7 & -1-14+15 & -7+10-3 \\ 10-3-7 & -2-21+5 & -14+15-1 \\ 15-1-14 & -3-7+10 & -21+5-2 \end{pmatrix} = \begin{pmatrix} -18 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -18 \end{pmatrix}$$

Test?:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \boxed{-2}\ \boxed{-3} \Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -5 & -2 & 1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right) \boxed{-1}$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & -5 & -7 & -3 & 0 & 1 \end{array}\right) \boxed{-2}\ \boxed{5} \Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 18 & 7 & -5 & 1 \end{array}\right) \boxed{\tfrac{1}{18}}$$

$$\Leftrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & -7 & -3 & 2 & 0 \\ 0 & 1 & 5 & 2 & -1 & 0 \\ 0 & 0 & 1 & \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{array}\right) \boxed{-5}\ \boxed{7} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right)$$

Cramers regel

Ex: Lös ekvationssystemet

$$\begin{cases} x + 2y + 3z = 2 \\ 2x + 3y + z = 5 \\ 3x + y + 2z = 1 \end{cases}$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\mathbf{x}} = \underbrace{\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}}_{b}$$

Eftersom $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ är inverterbar kan

vi lösa ekvationen $A\mathbf{x} = b$ genom att multiplicera till vänster med A^{-1}: $\mathbf{x} = A^{-1}A\mathbf{x} = A^{-1}b$

Vi kan sedan uttrycka A^{-1} med hjälp av adjunkten:

$$\mathbf{x} = \frac{1}{\det A}\begin{pmatrix} C_{11} & -C_{21} & C_{31} \\ -C_{12} & C_{22} & -C_{32} \\ C_{13} & -C_{23} & C_{33} \end{pmatrix}\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\det A}\begin{pmatrix} 2C_{11} - 5C_{21} + C_{31} \\ -2C_{12} + 5C_{22} - C_{32} \\ 2C_{13} - 5C_{23} + C_{33} \end{pmatrix}$$

kolonnutvecklingar

$$\boxed{= \frac{1}{\det A}\begin{pmatrix} \begin{vmatrix} 2 & 2 & 3 \\ 5 & 3 & 1 \\ 1 & 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & 1 \\ 3 & 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 5 \\ 3 & 1 & 1 \end{vmatrix} \end{pmatrix}} = -\frac{1}{18}\begin{pmatrix} 2\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 5\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 3 & 5 \end{vmatrix} \end{pmatrix}$$

$$= -\frac{1}{18}\begin{pmatrix} 2\cdot 5 - 5\cdot 1 + (-7) \\ -9 + 2\cdot 1 - 3\cdot(-13) \\ -2 - 2\cdot 0 + 3\cdot 4 \end{pmatrix} = -\frac{1}{18}\begin{pmatrix} -2 \\ 32 \\ 10 \end{pmatrix}$$

SVAR: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{18}\begin{pmatrix} -2 \\ 32 \\ 10 \end{pmatrix}$