The method just used can be modified so that angular sectors of the form $0 < \arg z < \alpha$, where α is not constrained to be $\pi/2$, can be mapped onto the interior of the unit circle (see Exercise 33 of this section). ●

8.4

EXERCISES

1. **a)** Derive Eq. (8.4–4) from Eq. (8.4–1).

b) Verify that Eq. (8.4–7) is equivalent to Eq. (8.4–1).

2. Suppose that the bilinear transformation (see Eq. 8.4–1) has real coefficients a, b, c, d. Show that a curve that is symmetric about the x-axis has an image under this transformation that is symmetric about the u-axis.

3. Derive Eq. (8.4–24) (the invariance of the cross ratio) by following the steps suggested in the text.

4. If a transformation $w = f(z)$ maps z_1 into w_1, where z_1 and w_1 have the same numerical value, we say that z_1 is a *fixed point* of the transformation.

a) For the bilinear transformation (Eq. (8.4–1) show that a fixed point must satisfy

$$cz^2 - (a - d)z - b = 0.$$

b) Show that unless $a = d \neq 0$ and $b = c = 0$ are simultaneously satisfied, there are at most two fixed points for this bilinear transformation.

c) Why are all points fixed points if $a = d \neq 0$ and $b = c = 0$ are simultaneously satisfied? Refer to Eq. (8.4–1).

Using the result of Exercise 4(a), find the most general form of the bilinear transformation $w(z)$ that has the following fixed points.

5. $z = -1$ and $z = 1$ **6.** $z = 1$ and $z = i$

For the transformation $w = 1/z$, what are the images of the following curves? Give the result as an equation in w or in the variables u and v, where $w = u + iv$.

7. $y = 1$ **8.** $x - y = 1$ **9.** $|z - 1 + i| = 1$ **10.** $|z + 1 + i| = \sqrt{2}$

11. $y = x$ **12.** $|z - 3 - 3i| = \sqrt{2}$ **13.** $|z - \sqrt{3} - i| = 1$

For the transformation $w = (z + 1)/(z - 1)$ what are the images of the following curves? Give the result as an equation in w or in the variables u and v.

14. $|z| = 1$ **15.** $|z| = 2$ **16.** $|z + 1| = 2$

Onto what domain in the w-plane do the following transformations map the domain $|z - 1| < 1$?

17. $w = \dfrac{z}{z - 1}$ **18.** $w = \dfrac{z - 1}{z}$ **19.** $w = \dfrac{z - 1}{(1 + i)z}$

Onto what domai
$1 < \text{Re}\, z < 2$?

20. $w = \dfrac{z}{z - 1}$

Find the bilinear tra
image points w_1, w

23. **a)** $z_1 = 0, z_2$
b) What is the

24. **a)** $z_1 = i, z_2 =$
b) What is the

25. **a)** $z_1 = \infty, z_2$
b) What is the

26. **a)** $z_1 = i, z_2 =$
b) What is the

27. The complex impe
generator of radia
$Z(\omega)$ progresses
line Re $Z = R$, Ir
$1/Z(\omega)$. Use the p
in the complex pla
and $Y(\infty)$.

28. **a)** A circle of radi
into another cir
show that this n

b) Is the image of
identical to the

c) Does the genera
circle in the z-pl

d) Consider the spe
ρ is mapped by t
$|a|\rho$. Thus in thi
images of each c

z-

Onto what domain in the w-plane do the following transformations map the domain $1 < \text{Re } z < 2$?

20. $w = \dfrac{z}{z-1}$ **21.** $w = \dfrac{z}{2z-3}$ **22.** $w = \dfrac{z-1}{z-2}$

Find the bilinear transformation that will map the points z_1, z_2, and z_3 into the corresponding image points w_1, w_2, and w_3 as described below:

23. a) $z_1 = 0,\ z_2 = i,\ z_3 = -i;\ w_1 = 1,\ w_2 = i,\ w_3 = 2 - i.$
b) What is the image of $|z| < 1$ under this transformation?

24. a) $z_1 = i,\ z_2 = -1,\ z_3 = -i;\ w_1 = 1 + i,\ w_2 = \infty,\ w_3 = 1 - i.$
b) What is the image of $|z| > 1$ under this transformation?

25. a) $z_1 = \infty,\ z_2 = 1,\ z_3 = -i;\ w_1 = 1,\ w_2 = i,\ w_3 = -i.$
b) What is the image of the domain $\text{Re}(z - 1) > \text{Im } z$ under this transformation?

26. a) $z_1 = i,\ z_2 = -i,\ z_3 = 1;\ w_1 = 1,\ w_2 = -i,\ w_3 = -1.$
b) What is the image of $|z| < 1$ under this transformation?

27. The complex impedance at the input of the circuit in Fig. 8.4–9 when driven by a sinusoidal generator of radian frequency ω is $Z(\omega) = R + i\omega L$. When ω increases from 0 to ∞, $Z(\omega)$ progresses in the complex plane from $(R, 0)$ to infinity along the semiinfinite line $\text{Re } Z = R$, $\text{Im } Z \geq 0$. The complex admittance of the circuit is defined by $Y(\omega) = 1/Z(\omega)$. Use the properties of the bilinear transformation to determine the locus of $Y(\omega)$ in the complex plane as ω goes from 0 to ∞. Sketch the locus and indicate $Y(0)$, $Y(R/L)$ and $Y(\infty)$.

28. a) A circle of radius $\rho > 0$ and center $(x_0, 0)$ is transformed by the inversion $w = 1/z$ into another circle. Locate the intercepts of the image circle on the real w-axis and show that this new circle has center $x_0/(x_0^2 - \rho^2)$ and radius $\rho/|x_0^2 - \rho^2|$.

b) Is the image of the center of the original circle under the transformation $w = 1/z$ identical to the center of the image circle? Explain.

c) Does the general bilinear transformation (see (Eq. 8.4–1)) always map the center of a circle in the z-plane into the center of the image of that circle in the w-plane? Explain.

d) Consider the special case of Eq. (8.4–1), $w = az + b$. Show that the circle $|z - z_0| = \rho$ is mapped by this transformation into a circle centered at $w_0 = az_0 + b$ with radius $|a|\rho$. Thus in this special case the original circle and its image have centers that are images of each other under the given transformation.

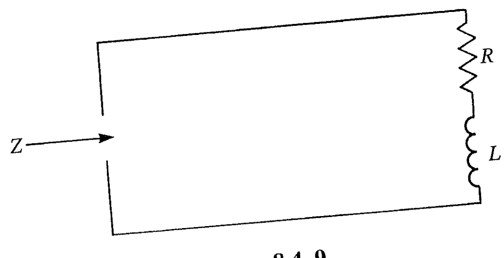

Figure 8.4–9