LECTURE 14
RESOLVENT SET AND SPECTRUM

Let \(G_A = \{(x, Ax), x \in D(A)\} \subset H \oplus H\) be the graph of the operator \(A\).

We define \(W : H \oplus H \to H \oplus H\) such that
\[
W(x, h) = (-x, h), \quad x, h \in H.
\]

Obviously \(W^2 = I\).

**Theorem 1.** If \(\overline{D(A)} = H\), then \((WG_A)^\perp = G_{A^*}\).

**Proof.** The fact that \((h, y) \in H \oplus H\) is orthogonal to \(WG_A\) implies
\[-(x, h) + (Ax, y) = 0.\]
Thus \((Ax, y) = (x, h)\) and by definition of the adjoint operator we have \(y \in D(A^*)\) and \(h = A^* y\). \(\square\)

**Theorem 2.** The operator \(A^*\) is always closed.

**Proof.** Indeed, the orthogonal complement to a linear subspace is always closed. Therefore \(G_{A^*} = (WG_A)^\perp\), which the graph of the operator \(A^*\), is a closed set. \(\square\)

**Theorem 3.** If \(\overline{D(A)} = H\). Then the condition \(\overline{D(A^*)} = H\) is equivalent to
\(A\) being closable. Moreover, in this case \(A^{**}\) exists and \(A^{**} = A\).

**Proof.** By Theorem 1 and since \(W^2 = I\) we obtain
\[
WG_{A^*} = W\left[WG_A\right]^\perp = \left[W^2G_A\right]^\perp = G_A.
\]

Therefore
\[
G_{A^{**}} = \left[WG_{A^*}\right]^\perp = G_A = G_{A^*}.
\]

\(\square\)

**Theorem 4.** The subspaces \(\overline{R(A)}\) and \(N(A^*)\) are orthogonal in \(H\) and
\[
H = \overline{R(A)} \oplus N(A^*).
\]

**Proof.** The element \(y \in N(A^*)\) if and only if \((Ax, y) = 0\) for all \(x \in D(A)\). This is equivalent to \(y \in R(A)\). \(\square\)
1. Spectrum and Resolvent of a Closed Operator

Let $A$ be a closed operator in a Hilbert space $H$.

**Definition.** $d_A = \text{def}A = \dim [R(A)]^\perp$ is called the defect of the operator $A$.

**Remark 1.** By using Theorem 4 we immediately obtain that $d_A = \dim N(A^*)$.

**Theorem 5.** Let us assume that $A$ is a closed operator such for some constant $C > 0$ $\|Ax\| \geq C\|x\|$ for all $x \in D(A)$. Let $B$ be an operator in $H$ such that $D(A) \subset D(B)$ and for any $x \in D(A)$

$$\|Bx\| \leq \alpha \|Ax\|, \quad \text{where}, \quad \alpha < 1.$$

Then

- $A + B$ is closed on $D(A)$
- $\|(A + B)x\| \geq (1 - \alpha)C\|x\|
- $d_{A+B} = d_A$.

**Proof.** The graph $G_A$ is closed w.r.t. the norm $|x|_A = \|x\| + \|Ax\|$. Therefore by using the triangle inequality we have

$$(1 - \alpha)|x|_A \leq |x|_{A+B} \leq (1 + \alpha)|x|_A.$$

Therefore the norms $|\cdot|_A$ and $|\cdot|_{A+B}$ — and thus since $G_A$ is closed then $G_{A+B}$ is closed. This implies that $A + B$ is closed. Now

$$\|(A + B)x\| \geq \|Ax\| - \|Bx\| \geq (1 - \alpha)\|Ax\| \geq (1 - \alpha)C\|x\|,$$

which proves the second statement of the theorem. Assume for a moment that $d_{A+B} < d_A$. Then there exists $f \in R(A)^\perp$, $f \neq 0$, such that $f \perp R(A + B)^\perp$. This implies $f \in R(A + B)$ and therefore there exists $y \in D(A)$ s.t. $f = (A + B)y$. Since $f \perp R(A)$ we have $(f, Ay) = 0$.

$$\|Ay\|^2 = (Ay, Ay) = -(By, Ay) \leq \|By\|\|Ay\| \leq \alpha\|Ay\|^2,$$

which gives a contradiction.

If we assume that $d_A + B > d_A$, then we can find $f = Ay$, $f \neq $ s.t. $f \perp R(A + B)$ and thus $(f, (A + B)y) = 0$. Finally

$$\|Ay\|^2 = (Ay, Ay) = -(By, Ay) \leq \|By\|\|Ay\| \leq \alpha\|Ay\|^2.$$

$\square$
Corollary 1. Let $A$ be an operator in $H$ satisfying the condition $\|Ax\| \geq C\|x\|$, $x \in D(A)$ and let $B$ be a bounded operator s.t. $\|B\| < C$. Then
\[ d_{A+B} = d_A. \]

Proof. $\|Bx\| \leq \|B\|\|x\| \leq \|B\|C^{-1}\|Ax\|$.
We now apply Theorem 5 with $a = \|B\|C^{-1}$.

Definition. The defect of the operator $A - \lambda I$ is denoted by $d_A(\lambda)$ and called the defect of $A$ at $\lambda$.

If $A - \lambda I$ has a bounded inverse on its image $(A - \lambda I)(H)$ (namely $\|(A - \lambda I)x\| \geq C\|x\|$ for some $C > 0$) then $\lambda$ is called a quasi-regular point of $A$.

All such points are denoted by $\beta(A)$.

Lemma 1. Let $A$ be a closed operator in $H$ such that $\|(A - \lambda I)x\| \geq C_0\|x\|$ for some $C_0 > 0$ and $\forall x \in D(A)$. Then
\[ D := \{ \lambda \in \mathbb{C} : |\lambda - \lambda_0| < C_0 \} \subset \beta(A) \]
and $d_A(\lambda)$ is constant on $D$.

Proof. If we write $A - \lambda I = (A - \lambda_0) + (\lambda_0 - \lambda)I$ then we complete the proof by using Corollary 1.

We now immediately obtain the following result:

Theorem 6. The set $\beta(A) \subset \mathbb{C}$ is open and the value of $d_A$ is constant on each connected component of $\beta(A)$.

Definition. If $d_A(\lambda) = 0$ for some $\lambda \in \beta(A)$, then $\lambda$ is called a regular point of $A$. In this case the operator $(A - \lambda I)^{-1}$ is bounded.

The set of all regular points of $A$ is called a resolvent set and denoted by $\rho(A)$.

Remark 2. The set $\rho(A)$ is open.

Definitions.
The set \( \sigma(A) = C \setminus \rho(A) \) is called the spectrum of the operator \( A \).

\( \hat{\sigma}(A) = C \setminus \hat{\rho}(A) \) is called the core of the spectrum.

The set \( \sigma_p(A) = \{ \lambda \in C : N(A - \lambda I) \neq \{0\} \} \) is called the point spectrum of \( A \) and \( \lambda \in \sigma_p \) is called the eigenvalue of \( A \).

The set \( \sigma_c(A) = \{ \lambda \in C : R(A - \lambda I) \neq R(A - \lambda I) \} \) is called the continuous spectrum of \( A \).

Example.
1. Let \( A = \frac{1}{i} \frac{d}{dt} \) defined on \( D(A) = \{ x : \int_{1}^{-1}(|x'(t)|^2 + |x(t)|^2) \, dt \} \subset L^2(-1, 1) \). \( D(A) \) is a dense in \( L^2(-1, 1) \) set. Solutions of the equation \( Ax = \lambda x \) are \( x(t) = e^{ikt} \) and \( \sigma_p(A) = k, k = 0, \pm 1, \pm 2, \ldots \).

2. Let \( A \) be defined as \( Ax(t) = tx(t) \) in \( L^2(0, \infty) \). The operator \( A \) is bounded and its spectrum is continuous and equal \( \sigma_c(A) = [0, \infty) \). (For the proof see Lecture 15).

Definition. Let \( \overline{D(A)} = H \). An operator \( A \) is called symmetric if \( (Ax, y) = (x, Ay) \forall x, y \in D(A) \).

It follows from the definition that \( A \subset A^* \). Therefore \( A \) can be closed and in particular \( \overline{A} = A^{**} \).

Home exercises.
1. Let \( A_0 = \frac{1}{i} \frac{d}{dt} \) defined the class of functions \( \{ x : x \in C_0^\infty(\mathbb{R}) \} \). Show that \( A_0 \) is symmetric and closable.

2. Let \( A = \frac{1}{i} \frac{d}{dt} \) defined on \( \{ x : \int_{1}^{-1}(|x'(t)|^2 + |x(t)|^2) \, dt \} \subset L^2(-1, 1) \) such that \( x(-1) = x(1) \). Show that \( A \) is self-adjoint.

3. Describe the closure of the class of functions \( \{ x : x \in C_0^\infty(\mathbb{R}) \} \) with respect to \( \int_{\mathbb{R}} |x'(t)|^2 \, dt \).