ON SHARP CONSTANTS IN LIEB-THIRRING INEQUALITIES

TIMO WEIDL
WEIDL@MATH.KTH.SE
TEL. 790 6663

In classical quantum mechanics the evolution of the wave function of a particle is described by the Schrödinger equation

$$\frac{1}{i}\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t),$$

where x is the spatial coordinate, t is the time and H is the Schrödinger operator. For the motion of a scalar particle in dimension one in an external potential $-V(x)$ this operator takes the form

$$H = -\frac{\partial^2}{\partial^2 x} - V(x).$$

Particles which remain trapped near the bottom of the potential well $-V(x)$ correspond to solutions

$$\psi_k(x,t) = e^{i\lambda_k t}\phi_k(x),$$

where ϕ_k are the eigensolutions with the eigenvalues λ_k of H, that is

$$(1) \qquad -\frac{\partial^2}{\partial^2 x}\phi_k(x) - V(x)\phi_k(x) = \lambda_k\phi_k(x), \qquad \int_{-\infty}^{\infty} |\phi_k(x)|^2 dx < \infty.$$

One of the main problems in the analysis of Schrödinger operators is to determine for a given real-valued function V the point spectrum σ_p of H, namely the set of λ_k, for which a solution ϕ_k in (1) exists. This problem can be solved explicitly only for very few potentials V. Therefore it is of great interest to deduce in the general case as much information on the point spectrum as possible.

We shall consider in particular potentials $V(x) \geq 0$, for which the integral

$$\int_{-\infty}^{\infty} V(x)dx$$

is finite. Then it is known, that the point spectrum forms a discrete subset of $(-\infty, 0]$. For a given $\lambda_k \in \sigma_p(H)$ there exists only $\mu_k < \infty$ linear independent solutions of (1), and the inequality [3]

$$(2) \qquad \frac{1}{4}\int_{-\infty}^{\infty} V(x)dx \leq \sum_{\lambda_k \in \sigma_p(H)} \mu_k |\lambda_k|^{1/2} \leq C\int_{-\infty}^{\infty} V(x)dx.$$

holds. This inequality is a special case in the scale of Lieb-Thirring estimates [2]. Bounds of this type are of great importance in proofs on the stability of matter.

The exact value of the constant C on the r.h.s. of (2) is so far unknown. In [2], [3] it has been shown, that

$$\frac{1}{2} \leq C < 1.005 \ .$$

It has been conjectured [2], that in fact $C = 1/2$. The proposal is to attack this problem for special potentials, which consists of $\delta-$functions

$$V(x) = -\sum_k \alpha_k \delta(x - x_k).$$

The work on the project will require

- to read on some basic facts in spectral theory,
- to find a theoretical proof, that for potentials consisting of two δ-functions the bound (2) holds with $C = 1/2$,
- to write a computer program, which calculates the point spectrum for potentials consisting of more than two δ-functions.

Preliminary knowledge in quantum mechanics is welcome but not necessary.

References

1. E. Fermi: "Notes on Quantum mechanics." *The University of Chicago Press*
2. E. Lieb, W. Thirring: "Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities." *Studies in Math. Phys., Essays in Honor of Valentine Bargmann.*, Princeton, (1976)
3. T. Weidl: "On the Lieb-Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$." *Communications in Mathematical Physics*, **178**, 1 (1996) 135-146.