Disability insurance: estimation and risk aggregation

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Introduction

- New upcoming regulation for insurance industry: Solvency II
- Stipulates methods for calculating capital requirements
- Standard model: scenario based approach
- Capital charge is given as the difference between
  - the present value of $A - L$ under best estimate assumptions
  - the present value in a certain shock scenario.

Introduction

- As an alternative, insurers may adopt an internal model
  - Should be based on a Value-at-Risk approach
- The capital charge is the difference between
  - the present value under best estimate assumptions
  - the p-quantile of the value in one year (here, $p = 0.005$).
- Disability rates fluctuate over time
  - Value-at-Risk dependent on future disability rates
  - Suggests us to consider stochastic disability models
- The aim of this talk:
  - suggest and fit a model for stochastic disability
  - determine systematic recovery risk for large portfolios in terms of p-quantiles

Introduction

This talk is based on the following four papers:

- Aro, H., Djehiche, B., and Löfdahl, B. (2015): Stochastic modelling of disability insurance in a multi-period framework. *Scandinavian Actuarial Journal*.
- Djehiche, B., and Löfdahl, B. (2014): A hidden Markov approach to disability insurance. *Preprint*.
- Djehiche, B., and Löfdahl, B. (2014): Risk aggregation and stochastic claims reserving in disability insurance. *Insurance: Mathematics and Economics*.
- Djehiche, B., and Löfdahl, B. (2015):Systematic disability risk in Solvency II. *Preprint.*

Disability inception

- Let $E_{x,t}$ denote the number of healthy individuals with age x at the beginning of year t
- Let $D_{x,t}$ denote the number of individuals among $E_{x,t}$ with disability inception in the interval $[t, t+1)$
- Assume $D_{x,t}$ is binomially distributed given $E_{x,t}$:

$$D_{x,t} \sim \mathrm{Bin}(E_{x,t}, p_{x,t})$$

where $p_{x,t}$ is the inception probability of an x-year-old.

Disability inception

- We suggest the following logistic regression model:

$$\mathrm{logit}\, p_{x,t} := \log\left(\frac{p_{x,t}}{1 - p_{x,t}}\right) = \sum_{i=1}^{n} \nu_t^i \phi^i(x),$$

  where $\phi^i(x)$ are age-dependent user-defined *basis functions* and ν_t^i are the model parameters for year t.

- Logistic regression is a widely used modelling tool in insurance, finance and many other areas.
  - The logistic transform guarantees that $p_{x,t} \in (0, 1)$.

Disability inception

Given historical values of $D_{x,t}$ and $E_{x,t}$, and a set of basis functions $\{\phi^i\}$, the log-likelihood function for yearly values of ν_t can be written

$$l(\nu_t; D_{\cdot,t}) = \sum_{x \in X} \left[D_{x,t} \sum_{i=1}^{n} \nu_t^i \phi^i(x) - E_{x,t} \log\left(1 + \exp\left\{\sum_{i=1}^{n} \nu_t^i \phi^i(x)\right\}\right)\right].$$

- If the basis functions are linearly independent, $-l(\nu_t)$ is strictly convex.
  - Unique estimate of ν_t
- Minimizing over $\mathbb{R}^n$ using methods from numerical optimization yields estimate of ν_t.
- Straightforward extension allows for modelling termination probabilities.

Modelling the future

The above models yield estimations for historical probabilities. What about the future?

Classical approach (Lee and Carter (1994), Aro and Pennanen (2011), Christiansen et. al. (2012) and others):

- Fit model to data, obtain time series of estimations of $\{\nu_t\}$
- Assume a stochastic process form for ν, estimate the parameters from this time series.

Inconsistent assumptions!

- ν_t is a parameter in the first step, realization of a stochastic process in the second step!
- May cause conceptual and numerical problems.

We suggest a hidden Markov model to perform both steps simultaneously.

Modelling the future

Assume ν unobservable Markov process with transition densities f parameterized by θ. Complete data likelihood becomes:

$$l(\theta; D_{\cdot,1:n}, \nu_{1:n}) = \sum_{t=1}^{n} \left[l(\nu_t; D_{\cdot,t}) + \log f_{\nu_t|\nu_{t-1}}(\theta) \right].$$

E-step: Given θ^k, integrate $l(\theta; D_{\cdot,1:n}, \nu_{1:n})$ with respect to the distribution of $\nu_{1:n}$ conditional on the observations $D_{\cdot,1:n}$, e.g. let

$$Q(\theta|\theta^k) = E^{\theta^k}[l(\theta; D_{\cdot,1:n}, \nu_{1:n})|D_{\cdot,1:n}],$$

M-step: maximize Q w.r.t. θ to obtain

$$\theta^{k+1} = \arg\max_{\theta} Q(\theta|\theta^k).$$

Numerical Results, disability inception

Assume ν multivariate Brownian motion with drift, i.e. let

$$\nu_t = \xi + \mu t + AW_t,$$

and let $\theta = (\xi, \mu, A)$.

- M-step can then be performed analytically.
- E-step requires particle simulation methods.
- We estimate θ and ν over the period from 2000-2011 using disability claims data from Folksam, with basis functions

$$\phi^1(x) = \frac{64 - x}{39} \quad \text{and} \quad \phi^2(x) = \frac{x - 25}{39}.$$

Numerical Results, disability inception

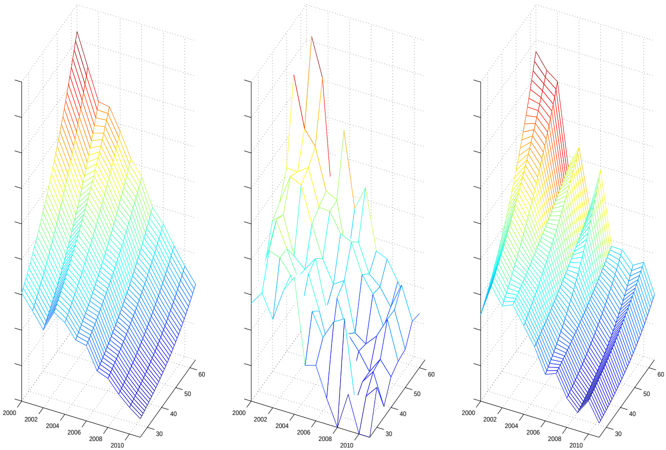

Figure: Center: Raw data. Right:Classical approach. Left:Hidden Markov.

Numerical Results, disability inception

Table: Relative difference of the estimated drift and volatility parameters between the two models.

	μ	σ
ν^1	0.92	0.48
ν^2	0.93	0.23

Continuous time modelling

We now bring a discrete termination model into a continuous time setting and consider

- claims reserving for annuity policies
- systematic recovery risk for large portfolios

A conditional independence model

Let Z be a stochastic process, let q_x be a non-negative function, and let $N^1, N^2, \ldots$ be counting processes starting from zero, with $\mathcal{F}^Z \vee \mathcal{F}^N$-intensities

$$\lambda_t^k = q_x(t, Z_{t^-})(1 - N_{t^-}^k). \tag{1}$$

- $N^1, N^2, \ldots$ represent the state of insured individuals
- Z represents the state of the economic-demographic environment
- x is a parameter representing eg. the age of the insured
- We assume that $N_t^1, N_t^2, \ldots$ are independent conditional on $\mathcal{F}_t^Z$.

Annuities and present values

The random present value of an annuity policy that pays $g_x(t, Z_t)$ monetary unit continuously as long as $N_t^k = 0$, until a fixed future time T_x, is given by

$$B_{t,T_x}^k = \int_t^{T_x} g_x(s, Z_s)(1 - N_s^k)e^{-\int_t^s r(u)du} ds, \qquad (2)$$

where r is the short rate.

- Allows for payments from the contract to depend on time, state of the economic-demographic environment and the age of the insured.
- For example, the contract could be inflation-linked and contain a deferred period.

Annuities and present values

Further, let X denote a finite set of age groups, and let $I_x^n,\ x \in X,\ n \geq 1$, denote the set of individuals with age x in a portfolio of n policies. Naturally, we must have

$$\sum_{x \in X} |I_x^n| = n. \tag{3}$$

Now, we consider a portfolio of annuity contracts. Define the portfolio random present value $B_t^{(n)}$ by

$$B_t^{(n)} = \sum_{\substack{x \in X \\ k \in I_x^n}} B_{t,T_x}^k. \tag{4}$$

Key idea: Take a large portfolio to diversify away the individuals. Only the systematic risk should remain.

Annuities and present values

If Z is Markov, the conditional expected value of the liabilities at time $t+1$, $L_{t+1}^{(n)}$, is given by

$$L_{t+1}^{(n)} = E[B_t^{(n)} | \mathcal{F}_{t+1}^N \vee \mathcal{F}_{t+1}^Z] = \sum_{\substack{x \in X \\ k \in I_x^n}} \left[B_{t,t+1}^k + (1 - N_{t+1}^k) e^{-\int_t^{t+1} r(u)du} v_x(t+1, Z_{t+1}) \right], \tag{5}$$

where

$$B_{t,t+1}^k = \int_t^{t+1} g_x(s, Z_s)(1 - N_s^k) e^{-\int_t^s r(u)du} ds, \tag{6}$$

$$v_x(t+1, Z_{t+1}) = E[\int_{t+1}^{T_x} g_x(s, Z_s) e^{-\int_{t+1}^s (q_x(u, Z_u) + r(u))du} ds | Z_{t+1}]. \tag{7}$$

Goal: determine the quantiles of $L_{t+1}^{(n)}$ at time t.

Large portfolios

The conditional Law of Large Numbers states that, conditional on $\mathcal{F}_t^N \vee \mathcal{F}_s^Z$ with $s \geq t$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} N_s^k - E[\frac{1}{n}\sum_{k=1}^{n} N_s^k | \mathcal{F}_t^N \vee \mathcal{F}_s^Z] = 0 \quad a.s. \tag{8}$$

Hence, using the conditional dominated convergence theorem, we have

$$\lim_{n\to\infty} \frac{1}{n} L_{t+1}^{(n)} - E[\frac{1}{n} L_{t+1}^{(n)} | \mathcal{F}_t^N \vee \mathcal{F}_{t+1}^Z] = 0 \quad a.s. \tag{9}$$

For a large portfolio, we suggest the approximation

$$L_{t+1}^{(n)} \approx E[L_{t+1}^{(n)} | \mathcal{F}_t^N \vee \mathcal{F}_{t+1}^Z]. \tag{10}$$

Large portfolios

Evaluating (10),

$$E[\frac{1}{n}L_{t+1}^{(n)}|\mathcal{F}_t^N \vee \mathcal{F}_{t+1}^Z] = \frac{1}{n}\sum_{\substack{x\in X\\ k\in I_x^n}}(1-N_t^k)V^x, \tag{11}$$

where

$$V^x = \int_t^{t+1} g_x(s,Z_s)e^{-\int_t^s \bar{q}_x(u,Z_u)du}ds + e^{-\int_t^{t+1}\bar{q}_x(u,Z_u)du}v_x(t+1,Z_{t+1}). \tag{12}$$

- First term represents benefits payed in $[t, t+1)$
- Second term represents value of remaining liabilities at $t+1$.
- What about the function v_x?

A reserve equation

Let $\bar{q}_x(t,z) = q_x(t,z) + r(t)$. Assume that $\bar{q}_x$ is lower bounded, g_x is continuous and bounded, and that Z is a Markov process with infinitesimal generator $\mathcal{A}$.

Then, $v_x(t,z)$ given by (7) satisfies the Feynman-Kac PDE

$$\begin{cases} -\frac{\partial v_x}{\partial s} + \bar{q}_x(s,z)v_x = \mathcal{A}v_x + g_x(s,z), \quad t \leq s < T_x, \\ v_x(T_x, z) = 0. \end{cases} \tag{13}$$

Classical but highly useful result:

- Value of remaining liabilities at $t+1$ can be calculated by solving (13)!

Large portfolios

For a large portfolio, we suggest the approximation

$$L_{t+1}^{(n)} \approx \sum_{\substack{x \in X \\ k \in I_x^n}} (1 - N_t^k) V^x. \tag{14}$$

- The idiosyncratic risk vanishes, only systematic risk remains!
- We approximate the portfolio quantiles by the computationally much simpler systematic risk quantile, obtainable by simulation of Z on $[t, t+1]$.
- It is hard to proceed further without simplifications.

Homogeneous portfolio and one-factor model

Consider a large, homogeneous portfolio under a one-factor model, let

$$V = \int_t^{t+1} g(s, Z_s) e^{-\int_t^s \bar{q}(u, Z_u) du} ds + e^{-\int_t^{t+1} \bar{q}(u, Z_u) du} v(t+1, Z_{t+1}), \tag{15}$$

so that

$$L_{t+1}^{(n)} \approx \sum_{k=1}^{n} (1 - N_t^k) V. \tag{16}$$

- 1-year risk still depends on Z on $[t, t+1]$.
- However, these types of liabilities tend to have a long duration (pensions, disability etc)
- The remaining liabilities value $v(t+1, Z_{t+1})$ will dominate.
- This motivates the use of a comonotonic approximation!

Homogeneous portfolio and one-factor model

Let the uniformly distributed random variable U be defined by

$$U = F_{Z_{t+1}}(Z_{t+1}), \tag{17}$$

define the stochastic process $\bar{Z}$ by

$$\bar{Z}_s := F_{Z_s}^{-1}(U) = F_{Z_s}^{-1}(F_{Z_{t+1}}(Z_{t+1})),\ s \leq t+1, \tag{18}$$

and define $\bar{V}$ by

$$\begin{aligned} \bar{V} = &\int_t^{t+1} g(s, \bar{Z}_s) e^{-\int_t^s \bar{q}(u, \bar{Z}_u) du} ds \\ &+ e^{-\int_t^{t+1} \bar{q}(u, \bar{Z}_u) du} v(t+1, Z_{t+1}). \end{aligned} \tag{19}$$

$\bar{V}$ is now simply a function of Z_{t+1}!

Homogeneous portfolio and one-factor model

Under reasonable assumptions on q, g and the generator $\mathcal{A}$ of Z, we can obtain a $\bar{V}$ that is monotone in Z_{t+1}!

For example, we can take $q > 0$ and increasing, $g > 0$ and deterministic, and define Z by

$$dZ_t = \alpha(t, Z_t)dt + \sigma(t, Z_t)dB_t. \tag{20}$$

Since $\bar{V}$ is then decreasing in Z_{t+1}, the sought quantiles are given by

$$\begin{aligned} F_{\bar{V}}^{-1}(p) = &\int_t^{t+1} g(s) e^{-\int_t^s \bar{q}(u, F_{Z_u}^{-1}(1-p))du} ds \\ &+ e^{-\int_t^{t+1} \bar{q}(u, F_{Z_u}^{-1}(1-p))du} v(t+1, F_{Z_{t+1}}^{-1}(1-p)). \end{aligned} \tag{21}$$

Numerical Results

We consider a disability recovery model fitted to data from Folksam, and simulate annuities paying 1 monetary unit continuously until recovery or the age of 65.

- Solve the Feynman-Kac PDE for $v(t+1,\cdot)$ on a large grid of z-values
- Simulate 10,000 paths of Z on $[t, t+1]$.
- For each path, simulate 2,000 contracts
- Examine convergence of the CLLN approximation
- Performance of the comonotonic approximation

Numerical Results, convergence

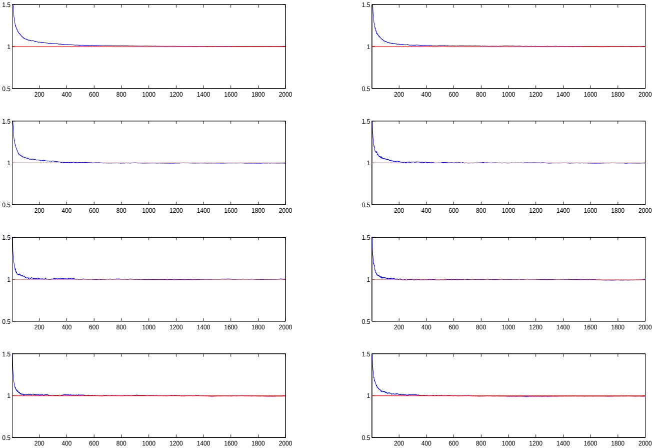

Figure: Simulation of 10,000 paths of Z to obtain 99,5% quantiles. Blue: Portfolio of up to 2,000 contracts. Red: CLLN approximation.

Numerical Results, comonotonic approximation

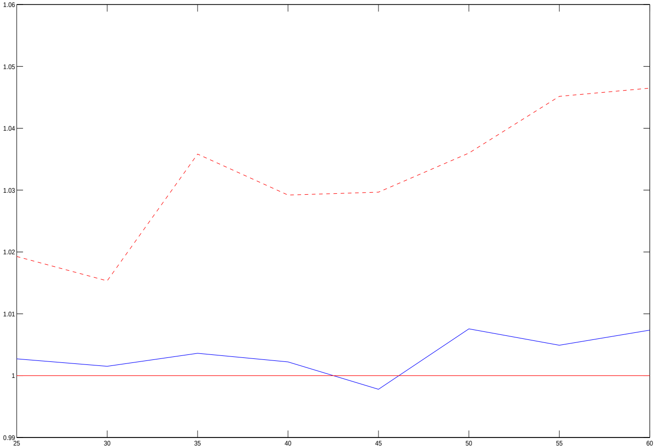

Figure: Simulation of 10,000 paths of Z to obtain 99,5% quantiles. Red: CLLN, Blue: 2,000 contracts, Red dashed: comonotonic approximation.

Extra: Approximations for multi-factor models

Claims termination often assumed to depend on age and disability duration.

- We propose the following logistic regression model:

$$\operatorname{logit} p_{x,d,t} = \sum_{i=1}^{n} \phi^i(x) \sum_{j=1}^{k} \nu_t^{ij} \psi^j(d),$$

  where ϕ^i and ψ^j, are age and duration dependent basis functions, respectively.
- We can fit the model parameters using the EM-algorithm.
- How can we use this model to calculate one-year risks?

Extra: Approximations for multi-factor models

Define the process Z by

$$Z_t = \sum_{i=1}^{n} \phi^i(x) \sum_{j=1}^{m} \psi^j(t) \nu_t^{i,j} =: a(t)^T \nu_t. \tag{22}$$

A continuous time approximation of the logistic regression model yields the intensity

$$q(t, \nu_t) = c \log \left(1 + \exp\left\{Z_t\right\}\right) =: f(Z_t). \tag{23}$$

- Z is scalar valued, so if we can find the generator of Z, we have a one-factor model!
- However, in general, Z need not even be Markov.

Extra: Approximations for multi-factor models

From the Itô formula,

$$dZ_t = (\dot{a}(t)^T \nu_t + a(t)^T \mu)dt + a(t)^T A dW_t, \tag{24}$$

which cannot directly be written as

$$dZ_t = \alpha(t, Z_t)dt + \gamma(t)d\bar{W}_t. \tag{25}$$

To obtain a process $\widehat{Z}$ of the form (25), we consider the Markov projection technique introduced by Krylov (1984) and extended in various ways by Gyöngy (1986), Kurtz and Stockbridge (1998) and others.

If we choose α s.t.

$$\alpha(t, z) = E[\dot{a}(t)^T \nu_t + a(t)^T \mu | Z_t = z], \tag{26}$$

then $\widehat{Z}$ and Z have the same marginal distributions.

Extra: Approximations for multi-factor models

We obtain the following explicit expression for α:

$$\alpha(t, z) = a^T \mu + \dot{a}^T(\xi + \mu t) + (z - a^T(\xi + \mu t))\frac{a^T A A^T \dot{a}}{a^T A A^T a} \quad (27)$$

Curiously, it happens that $\widehat{Z}$ is a Hull-White process.

The quantiles of $\widehat{Z}_s$ are given by

$$F^{-1}_{\widehat{Z}_s}(1-p) = a(s)^T(\xi + \mu s) + \sqrt{s a(s)^T A A^T a(s)}\,\Phi^{-1}(1-p). \quad (28)$$

These quantiles are plugged into the comonotonic approximation formula!

Equality of marginal distributions does not imply equality of systematic risk quantiles, but quantitative studies suggest the error is in the order of 1%.

Thank you for your attention!