Conjugate duality in stochastic optimization

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Introduction

We study convex stochastic optimization problems.

- ✔ Stochastic LP duality, linear quadratic control and calculus of variations
- ✔ Stochastic problems of Bolza, shadow price of information and optimal stopping
- ✔ Illiquid convex market models (Jouni&Kallal, Kabanov, Schachermayer, Guasoni, Pennanen)
- ✔ Super-hedging and pricing, utility maximization and optimal consumption

Convexity gives rise to dual optimization problems and dual characterisations of the objective functionals.

Introduction

Example (Super-hedging in a liquid market)**.**

$$\inf x_0^0,$$
$$\textit{s.t.} \left\{ \quad C_T \le x_0^0 + \int_T x_t \cdot dS_t \ \textit{a.s.} \right.$$

where x_0^0 is the initial wealth, C_T is a claim, x is a predictable process (portfolio of risky assets) and S is a price process. The infimum is over initial wealths x_0 and predictable processes x.

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The dual problem is

$$\sup_{Q \in \mathcal{M}} \mathbb{E}^Q[C_T],$$

where the supremum is over martingale measures.

Introduction

Example (Kabanovs model). *Consider a set*

$$\{x \in BV \,|\, (dx/|dx|)_t \in C(\omega, t) \ \forall t\}$$

where $C(\omega, t) \subset \mathbb{R}^d$ *is a convex cone for all* (ω, t). $C(\omega, t)$ *is the set of self-financing trades in the market at time* t. *A predictable process of bounded variation is self-financing if*

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Example (Linear case).

$$C(\omega, t) = \{(x^0, x^1) \in \mathbb{R}^2 \,|\, x^0 + x^1 \cdot S_t(\omega) \leq 0\},$$

where S is the price process of a risky asset, x^0 refers to a bank account and x^1 to the risky asset. Portfolio is self-financing if all trades of the risky assets are financed using the bank account. The inequality allows a free disposal of money or assets.

Example (Optimal consumption in a convex market model)**.**

$$\sup \mathbb{E} \int_T U_t(\omega, dc),$$

$$\text{s.t.} \begin{cases} (d(x(\omega)+c(\omega))/|d(x(\omega)+c(\omega))|)_t \in C(\omega,t) \ \forall t \text{ a.s.} \\ x_t(\omega) \in D(\omega,t) \ \forall t \text{ a.s.}. \end{cases}$$

where U_t is an utility function for all t almost surely and $D(\omega,t)$ is the set of allowed portfolio positions at time t. The supremum is over predictable processes of bounded variation x and c. x is the portfolio process, and c is the consumption process.

Example (Optimal consumption in a convex market model)**.** *A dual problem is*

$$\inf \mathbb{E} - \int_T U_t^*(y_t)dt,$$

$$\textit{s.t.} \begin{cases} y_t(\omega) \in C^*(\omega, t) \ \forall t \ \textit{a.s.} \\ (da(\omega)/|da(\omega)|)_t \in D^*(\omega, t) \ \forall t \ \textit{a.s..} \end{cases}$$

where U_t^* is the concave conjugate of the utility function, $C^*(\omega, t)$ is the polar of $C(\omega, t)$ (y is a consistent price system*), $D^*(\omega, t)$ is the polar of $D(\omega, t)$, and the supremum is over semimartingales with the canonical decomposition $y = m + a$.*

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The aim is to formulate problems like this in a general framework and deduce the dual problems by general methods.

Stochastic Optimization

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$$F(x, u) = \mathbb{E}[f(\omega, x(\omega), u(\omega))],$$

where f is a normal-integrand. The *value function* is

$$\phi(u) = \inf_{x \in \mathcal{N}} F(x, u) = \inf_{x \in \mathcal{N}} \mathbb{E}[f(\omega, x(\omega), u(\omega))].$$

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A function $f : \Omega \times (X \times U) \to \mathbb{R} \cup \{+\infty\}$ is a normal integrand if the epigraph $\operatorname{epi} f \subset \Omega \times X \times U \times \mathbb{R}$ is measurable and ω-sections are closed.

In particular $\omega \mapsto f(\omega, x(\omega), u(\omega))$ is measurable when $x \in L^0(\Omega; X)$ and $u \in L^0(\Omega; U)$, and for fixed ω, $(x, u) \mapsto f(\omega, x, u)$ is lower semicontinuous. Moreover, F is convex if $f(\omega, \cdot, \cdot)$ is convex.

Example (Convex market models). *Let $U = BV$, and*

$$f(\omega, x, u) = k(\omega, x, u) + \delta_{\mathcal{D}(\omega)}(x) + \delta_{\mathcal{C}(\omega)}(dx + du),$$

where (and similarly for $\delta_{\mathcal{C}(\omega)}$)

$$\delta_{\mathcal{D}(\omega)}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{D}(\omega) \\ +\infty & \text{otherwise,} \end{cases}$$

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and k is a normal integrand which gives the criterion one wants to minimize/maximize (e.g. utility). In this case $u \in L^p(\Omega; U)$ can be interpreted as a claim process or a consumption process.

Conjugate duality

Let Y be a separable Banach space and U be its dual space. The pairing $\langle u, y\rangle = \mathbb{E}\langle u(\omega), y(\omega)\rangle$ is finite for all $u \in L^p(\Omega; U)$ and $y \in L^q(\Omega; Y)$. We equip these spaces with weak topologies induced by the pairing.

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The *convex conjugate* of $\phi : L^p(\Omega; U) \to \mathbb{R} \cup \{\pm\infty\}$ is defined by

$$\phi^*(y) = \sup_{u \in L^p(\Omega; U)} \{\langle u, y\rangle - \phi(u)\},$$

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The biconjugate satisfies $\phi^{**} = \operatorname{cl} \operatorname{co} \phi$, where

$$\operatorname{cl} \phi = \begin{cases} -\infty & \text{if } \operatorname{lsc} \phi(u) = -\infty \text{ for some } u, \\ \operatorname{lsc} \phi & \text{otherwise.} \end{cases}$$

In particular, if ϕ is convex and closed, then $\phi = \phi^{**}$ (the *dual representation*).

Conjugate duality

Recall the value function $\phi : L^p(\Omega; U) \to \mathbb{R} \cup \{\pm\infty\}$ was given by

$$\phi(u) = \inf_{x \in \mathcal{N}} \mathbb{E} f(x(\omega), u(\omega)),$$

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which is a convex function on $\mathcal{U}$ (if F is convex, which we assume).

Define the *dual objective* by

$$g(y) = -\phi^*(y),$$

which is a concave upper semicontinuous function on $\mathcal{Y}$. If ϕ is lower semicontinuous and proper, then ϕ has the dual representation

$$\phi(u) = \sup_{y \in L^q(\Omega; Y)} \{\langle u, y \rangle + g(y)\}.$$

Conjugate duality in stochastic optimization

$$\phi(u) = \inf_{x \in \mathcal{N}} \mathbb{E} f(\omega, x(\omega), u(\omega)).$$

- ✔ Calculating the dual objective $g = -\phi^*$ is based on conjugacy of integral functionals and theory of normal-integrands. Adaptiveness constraints lead to stochastic analysis; also the dual problem may not be a pure integral functional anymore.

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- ✔ Calculating the dual objective $g = -\phi^*$ is based on conjugacy of integral functionals and theory of normal-integrands. Adaptiveness constraints lead to stochastic analysis; also the dual problem may not be a pure integral functional anymore.
- ✔ In convex analysis there exists a lot of results for the lower semicontinuity of ϕ. These are based on LCTVS structure of the strategy space and compactness type arguments.
- ✔ $\mathcal{N}$ is not LCTVS. In mathematical finance there exists results for lower semicontinuity of ϕ in this case, but only when $f(\omega, x, u)$ is an indicator function or of some other very restrictive form.

Conjugate duality in stochastic optimization

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Example. *Assume there exists (v, y) such that $\mathbb{E}f^*(\omega, v(\omega), y(\omega)) < \infty$, v is a martingale, and*

$$\{x \in X \,|\, \exists u \in B(\omega),\ f(\omega, x, u(\omega)) - \langle x, v(\omega)\rangle \leq \beta(\omega)\}$$

is compact almost surely for some $\beta \in L^0(\Omega; \mathbb{R})$, where $B(\omega)$ is a neighborhood of the origin almost surely. Then the value function ϕ is lower semicontinuous at the origin.

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Remark. *The path spaces X, U, Y can be generalized to Souslin LCTVS, and perturbation space $L^p(\Omega; U)$ can be generalized to LCTVS. Banach space structure shown in the slides was just an example.*

Conjugate duality in stochastic optimization

And back to introduction: many convex stochastic optimization problems are covered by this duality framework.

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- ✔ Stochastic problems of Bolza, shadow price of information and optimal stopping
- ✔ Illiquid convex market models
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