Anisotropic Hastings-Levitov clusters and harmonic measure

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March 16, 2010

Joint work with Fredrik Johansson (KTH) and Amanda Turner (Lancaster).

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Distributions of d_n and θ_n to be specified in a moment.

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We get a sequence of growing compacts: $K_n = \mathbb{C} \setminus f_n(\Delta)$.

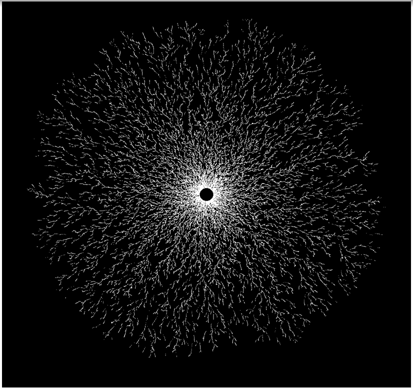

Figure: Cluster with slit lengths $d_j = d_0$ fixed, angles uniform in $[0, 2\pi]$. 25.000 iterations

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Solution: choose new length as

$$d_n = \frac{\delta_0}{|f'_{n-1}(e^{i\theta_n})|^{\alpha/2}}, \quad 0 \leq \alpha \leq 2.$$

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Some rigorous results (for a regularized version) due Rohde and Zinsmeister (2005), recent work by Norris and Turner (2008) in the case $\alpha = 0$
Related work by Johansson and S. (Loewner chains driven by compound Poisson processes)

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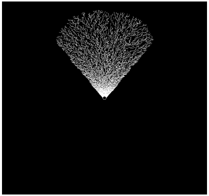
- $d\nu = \chi_{[0,\eta]}(\theta) d\theta$, for some $\eta \in (0, 2\pi)$
- $d\nu = \cos^2(m\theta) d\theta$, for $m = 1, 2, \ldots$.



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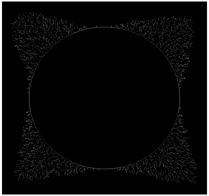

Figure: Cluster with slit lengths $d_j = d_0$ fixed, angles with density $\cos^2(mx)$. 25.000 iterations

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THM (Johansson, S, Turner): Then, as $d \to 0$, we have

$$X_d \to \psi \qquad (3)$$

in probability in the Skorokhod space.

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- use Grönwall's lemma, and additional estimates

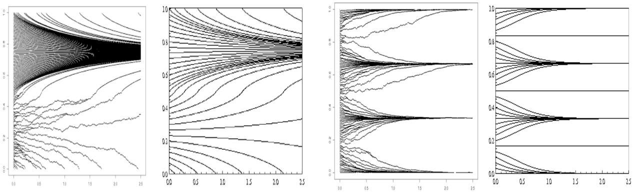
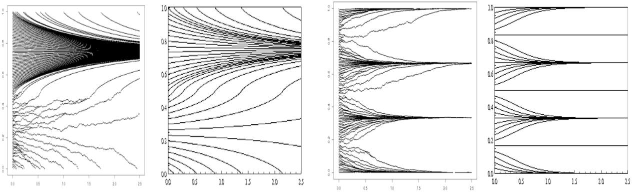


Figure: Left: the process Z_d, right: the limit flow.



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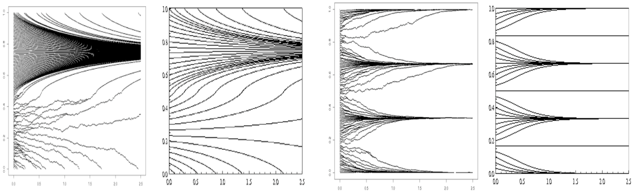

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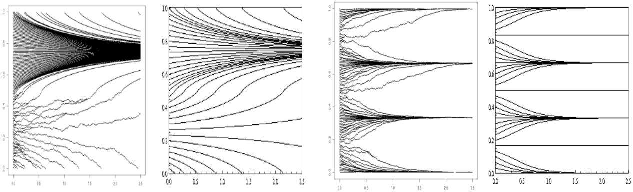

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- positions at which fingers are rooted to unit disk converge to unstable equilibria of the ODE.

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in distribution.The proof is somewhat technical.

Worth pointing out:

- Anisotropy kills randomness in the scaling limit (cf. Brownian web in the isotropic case-pure randomness!).
- No fluctuations outside the support of ν.
- A global description of the clusters in terms of conformal mappings and the Loewner equation is possible.
- Can allow for random slit lengths, tending to 0 in an appropriate sense.
- Dependence in slit lengths still hard to handle—future work.

References:

- M.B. Hastings, L. Levitov, Laplacian growth as 1d turbulence, Physica D (1998)
- S. Rohde, M. Zinsmeister, Some remarks on Laplacian growth, Topology Appl. (2005)
- J. Norris, A. Turner, Planar aggregation and coalescing Brownian flow, arXiv 2008
- F. Johansson, A. Sola, A. Turner, Scaling limits of anisotropic Hastings-Levitov clusters, arXiv 2009.

Thank you for your attention!