Full-information best-choice game with two stops

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Best-choice problem

- \( N \) i.i.d. random variables from a known distribution function \( F(x) \) are observed sequentially with the object of choosing the largest.

- At the each stage observer should decide either to accept or to reject the variable.

- Variable rejected cannot be considered later.

- The aim is to maximize the expected value of the accepted variable.

Let \( F(x) \) is uniform on \([0, 1]\).

The threshold strategy satisfies the equation (Mozer’s equation):

\[
v_i = \frac{1 + v_{i+1}^2}{2}, \quad i = 1, 2, ..., N - 1, \quad v_N = 1/2.
\]
Optimal stopping problem:

j.P. Gilbert and F. Mosteller (1966), L. Mozer (1956)
E.B. Dynkin and A.A. Yushkevich (1967)

Game-theoretic approach:

M. Sakaguchi
V. Baston and A. Garnaev (2005)
A. Garnaev and A. Solovyev (2005)
M. Sakaguchi and V. Mazalov
K. Szajowski (1992)

Problem with two stops:

G. Sofronov, J. Keith, D. Kroese (2006)
M. Sakaguchi (2003)
$m$-person best-choice game with one stop

- Each of $m$ companies (players) wants to employ a secretary among $N$ applicants.

- Each player observes the value of applicant’s quality and decides either to accept or to reject the applicant.

- Applicants’ qualities have uniform distribution on [0,1].

- If the player $j$ accepts an applicant then there is probability $p_j$ that the applicant rejects the proposal, $j = 1, 2, ..., m$.

- If player $j$ employs a secretary then he leaves the game. The payoff of the player is equal to the expected quality’s value of selected secretary.

- Applicant rejected by player cannot be considered later.

- The shortfall of a player not employing an applicant is $C$, $C \in [0, 1]$.

- Each player aims to maximize his expected payoff.
One player

\[ \bar{p}_1 = 1 - p_1. \]

\[ v_i^1(p_1) \] – expected payoff of the player at the stage \( i, i = 1, 2, ..., N. \)

\[
v_N^1(p_1) = \int_0^1 p_1 x \, dx + \int_0^1 \bar{p}_1 (-C) \, dx = \frac{p_1}{2} - \bar{p}_1 C.
\]

The player accepts the \( i \)-th applicant with quality value \( x \) if \( x \geq v_{i+1}^1(p_1) \).

\[
v_i^1(p_1) = \mathbb{E}(\max \{ p_1 x + \bar{p}_1 v_{i+1}^1(p_1); v_{i+1}^1(p_1) \})
\]

\[
= \frac{p_i}{2} (1 - v_{i+1}^1(p_1))^2 + v_{i+1}^1(p_1),
\]

\[ v_{N+1}^1(p_1) = -C, i = 1, 2, ..., N. \]

Table 1. Optimal thresholds for \( N = 10, p_1 = 0, C = 0. \)

<table>
<thead>
<tr>
<th>( i )</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>( v_i^1(p_1) )</td>
<td>0.850</td>
<td>0.836</td>
<td>0.820</td>
<td>0.800</td>
<td>0.775</td>
<td>0.742</td>
<td>0.695</td>
<td>0.625</td>
<td>0.5</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Two players (A. Garnaev, A. Solovyev, 2005)

The expected payoff of the \( j \)-th player at the stage \( i \) is \( v_{i}^{2,j}, j = 1, 2, i = 1, ..., N \).

\[ v_{N}^{2,j} = v_{N}^{1}(p_{j}), j = 1, 2. \]

At the stage \( N - 1 \) the matrix of the game is following:

\[
M_{N-1}^{2}(x) = \begin{pmatrix}
A_{2} & R_{2} \\
A_{1} & R_{1}
\end{pmatrix}
\begin{pmatrix}
(m_{11}^{1}, m_{11}^{2}) & (m_{12}^{1}, m_{12}^{2}) \\
(m_{21}^{1}, m_{21}^{2}) & (m_{22}^{1}, m_{22}^{2})
\end{pmatrix},
\]

where

\[
\begin{align*}
m_{11}^{1} &= p_{1}x + v_{N}^{1}(p_{1}) + p_{2}v_{N}^{1}(p_{1}) + (1 - p_{1} - p_{2})v_{N}^{2,1}; \\
m_{11}^{2} &= p_{2}x + p_{1}v_{N}^{1}(p_{2}) + (1 - p_{1} - p_{2})v_{N}^{2,2}; \\
m_{12}^{1} &= p_{1}x + v_{N}^{2,1} + (1 - p_{1})v_{N}^{2,1}; \\
m_{12}^{2} &= p_{1}v_{N}^{1}(p_{2}) + p_{1}v_{N}^{2,1}; \\
m_{21}^{1} &= p_{2}v_{N}^{1}(p_{1}) + p_{2}v_{N}^{2,1}; \\
m_{21}^{2} &= p_{2}x + p_{2}v_{N}^{2,2}; \\
m_{22}^{1} &= v_{N}^{2,1}; \\
m_{22}^{2} &= v_{N}^{2,2}.
\end{align*}
\]

\[
v_{i}^{2,j} = \int_{0}^{v_{i+1}^{2,j}} v_{i+1}^{1} dx + \int_{v_{i+1}^{2,j}}^{1} (p_{j}x + p_{j}v_{i+1}^{2,j}) dx = v_{i}^{1}(p_{j}); j = 1, 2.
\]
\textbf{m players}

The expected payoff of the \( j \)-th player at the stage \( i \) is \( v_{i,j}^m, \; j = 1, 2, \ldots, m, \; i = 1, \ldots, N \).

The player \( j \) accepts the \( i \)-th applicant with quality value \( x \) if

\[ x \geq v_{i+1}^{m,j}, \; i = 1, 2, \ldots, N - 1. \]

\textbf{Theorem 1} \textit{In the \( m \)-person best-choice game each player uses an optimal strategy as if the other players were not there, that is, \( v_{i,j}^m = v_i^1(p_j), \; j = 1, 2, \ldots, m; \; i = 1, \ldots, N - 1; \; v_N^1(p_j) = \frac{p_j}{2} + \bar{p}_j C \) for every \( m \).}
$m$-person best-choice game with two stops

- Each of $m$ companies (players) wants to employ two secretaries among $N$ applicants.

- Each player observes the value of applicant’s quality and decides either to accept or to reject the applicant.

- Applicants’ qualities have uniform distribution on $[0,1]$.

- If player $j$ accepts an applicant then there is probability $p_j$ that the applicant rejects the proposal $j = 1, 2, ..., m$.

- If player $j$ employs two secretaries then he leaves the game. The payoff of the player is equal to sum of the expected quality values of selected secretaries.

- Applicant rejected by player cannot be considered later.

- The shortfall of a player not employing any applicant is $C$, $C \in [0,1]$.

- Each player aims to maximize his expected payoff.
One player

\( v_i^1(p_j) \) — expected payoff of the player at the stage \( i \)

\( v_{i,r}^1(p_j) \) — expected payoff of the player at the stage \( r \) on condition he has already employed a secretary at the stage \( i \)

The expected player’s payoff if he stays in the game alone is following

\[
\begin{align*}
  v_i^1(p_j) &= \mathbb{E}\left( \max \left\{ p_j (X_i + v_{i,i+1}^1(p_j)) + \bar{p}_j v_{i+1}^1(p_j); v_{i+1}^1(p_j) \right\} \right), \ i = 1, 2, ..., N, \\
v_{N+1}^1(p_j) &= -C; \\
v_{i,r}^1(p_j) &= \mathbb{E}\left( \max \left\{ p_j X_r + \bar{p}_j v_{i,r+1}^1(p_j); v_{i,r+1}^1(p_j) \right\} \right), \ r = i + 1, ..., N, \\
v_{i,N+1}^1(p_j) &= -C.
\end{align*}
\]

If the player has already employed an applicant at the stage \( i \), he accepts another applicant if \( x \geq v_{i,r+1}^1(p_j) \).

The first applicant would be accepted at the stage \( i \) if \( x \geq v_{i+1}^1(p_j) - v_{i,i+1}^1(p_j) \).
\[
v_i^1 = v_{i,i+1}^1 + \int_0^{v_{i+1}^1 - v_{i,i+1}^1} (v_{i+1}^1 - v_{i,i+1}^1) dx + \int_{v_{i+1}^1 - v_{i,i+1}^1}^1 (p_j x + \bar{p}_j (v_{i+1}^1 - v_{i,i+1}^1)) dx \\
= v_{i+1}^1 + \frac{p_j}{2} (1 - (v_{i+1}^1 - v_{i,i+1}^1))^2;
\]
\[
v_{i,r}^1 = \int_0^{v_{i,r+1}^1} v_{i,r+1}^1 dx + \int_{v_{i,r+1}^1}^1 (p_j x + (1 - p_j) v_{i,r+1}^1) dx = v_{i,r+1}^1 + \frac{p_j}{2} (1 - v_{i,r+1}^1)^2;
\]
\[
v_{i,N}^1 = \frac{p_j}{2} - \bar{p}_j C;
\]
\[
v_{i,r}^1 = v_{i,r}^1 (p_j); v_i^1 = v_i^1 (p_j), i = 1, ..., N - 1, r = i + 1, ..., N.
\]

Table 2. Optimal thresholds for \( N = 10, p_j = 0, C = 0 \)

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<tr>
<td>( v_{i+1}^1 - v_{i,i+1}^1 )</td>
<td>0.757</td>
<td>0.735</td>
<td>0.708</td>
<td>0.676</td>
<td>0.634</td>
<td>0.579</td>
<td>0.5</td>
<td>0.375</td>
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Two players

$v^2_{i,j}$ — expected payoff of the $j$-th player at the stage $i$

$v^2_{i,r}, j = 1, 2$ — expected payoff of the $j$-th player at the stage $r$ on condition he has already employed a secretary at the stage $i$

At the stage $N - 2$ if the first player hasn’t employed a secretary and the second player selected one, the matrix of the game is as following:

$$M^2_{N-2}(x) = \begin{pmatrix} A_1 & A_2 \\ R_1 & R_2 \end{pmatrix} \begin{pmatrix} (m^1_{11}, m^2_{11}) \\ (m^1_{21}, m^2_{21}) \end{pmatrix},$$

where

$$
\begin{align*}
m^1_{11} &= p_1(x + v^2_{i,N-1}) + p_2v^1_{N-1}(p_1) + (1 - p_1 - p_2)v^2_{N-1}; \\
m^2_{11} &= p_2x + p_1v^2_{i,N-1} + (1 - p_1 - p_2)v^2_{N-1}; \\
m^1_{12} &= p_1(x + v^2_{N-2,N-1}) + (1 - p_1)v^2_{N-1}; \\
m^2_{12} &= p_1v^1_{i,N-1}(p_2) + \bar{p}_1v^2_{i,N-1}; \\
m^1_{21} &= p_2v^1_{N-1}(p_1) + \bar{p}_2v^2_{N-1}; \\
m^2_{21} &= p_2x + \bar{p}_2v^2_{i,N-1}; \\
m^1_{22} &= v^2_{N-1}; \\
m^2_{22} &= v^2_{i,N-1}.
\end{align*}
$$
\textit{m-person game}

\( v^{m,j}_i, j = 1, 2, ..., m \) — expected payoff of the \( j \)-th player at the stage \( i \)

\( v^{m,j}_{i,r}, j = 1, 2, ..., m \) — expected payoff of the \( j \)-th player at the stage \( r \) on condition he has already employed a secretary at the stage \( i \)

**Theorem 2** in the \( m \)-person best-choice game each player uses an optimal strategy as if the other players were not there, that is, \( v^{m,j}_i = v^{1}_i(p_j), i = 1, ..., N - 1; \)
\( v^{m,j}_{i,r} = v^{1}_{i,r}(p_j), r = i + 1, ..., N; v^{1}_{i,N}(p_j) = \frac{p_j}{2} + \overline{p}_j C, j = 1, 2, ..., m. \)
References


