Full-information best-choice game with two stops

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Best-choice problem

- N i.i.d. random variables from a known distribution function $F(x)$ are observed sequantially with the object of choosing the largest.
- At the each stage observer should decide either to accept or to reject the variable.
- Variable rejected cannot be considered later.
- The aim is to maximize the expected value of the accepted variable.

Let $F(x)$ is uniform on $[0, 1]$.

The threshold strategy satisfies the equation (Mozer's equation):

$$v_i = \frac{1 + v_{i+1}^2}{2}, i = 1, 2, ..., N - 1, v_N = 1/2.$$

Optimal stopping problem:

j.P. Gilbert and F. Mosteller (1966), L. Mozer (1956)

E.B. Dynkin and A.A. Yushkevich (1967)

Game-theoretic approach:

M. Sakaguchi

V. Baston and A. Garnaev (2005)

A. Garnaev and A. Solovyev (2005)

M. Sakaguchi and V. Mazalov

K. Szajowski (1992)

Problem with two stops:

G. Sofronov, J. Keith, D. Kroese (2006)

M. Sakaguchi (2003)

M.L. Nikolaev (1998)

m-person best-choice game with one stop

- Each of m companies (players) wants to employ a secretary among N applicants.
- Each player observes the value of applicant's quality and decides either to accept or to reject the applicant.
- Applicants' qualities have uniform distribution on [0,1].
- If the player j accepts an applicant then there is probability p_j that the applicant rejects the proposal, $j = 1, 2, ..., m$.
- If player j employs a secretary then he leaves the game. The payoff of the player is equal to the expected quality's value of selected secretary.
- Applicant rejected by player cannot be considered later.
- The shortfall of a player not employing an applicant is C, $C \in [0, 1]$.
- Each player aims to maximize his expected payoff.

One player

$\bar{p}_1 = 1 - p_1$.

$v_i^1(p_1)$ – expected payoff of the player at the stage i, $i = 1, 2, ..., N$.

$$v_N^1(p_1) = \int_0^1 p_1 x\, dx + \int_0^1 \bar{p}_1(-C)dx = \frac{p_1}{2} - \bar{p}_1 C.$$

The player accepts the i-th applicant with quality value x if $x \geq v_{i+1}^1(p_1)$.

$$\begin{aligned}
&v_i^1(p_1) = \mathbf{E}(\max\left\{p_1 x + \bar{p}_1 v_{i+1}^1(p_1); v_{i+1}^1(p_1)\right\}) \\
&= \tfrac{p_1}{2}(1 - v_{i+1}^1(p_1))^2 + v_{i+1}^1(p_1), \\
&v_{N+1}^1(p_1) = -C, i = 1, 2, ..., N.
\end{aligned}$$

Table 1. Optimal thresholds for $N = 10$, $p_1 = 0$, $C = 0$.

i	1	2	3	4	5	6	7	8	9	10
$v_{i+1}^1(p_1)$	0.850	0.836	0.820	0.800	0.775	0.742	0.695	0.625	0.5	0

Two players (A. Garnaev, A. Solovyev, 2005)

The expected payoff of the j-th player at the stage i is $v_i^{2,j}, j = 1, 2, i = 1, ..., N$.

$$v_N^{2,j} = v_N^1(p_j), j = 1, 2.$$

At the stage $N-1$ the matrix of the game is following:

$$M_{N-1}^2(x) = \begin{array}{c} \\ \mathsf{A}_1 \\ \mathsf{R}_1 \end{array} \begin{array}{cc} \mathsf{A}_2 & \mathsf{R}_2 \\ \left(\begin{array}{c} (m_{11}^1, m_{11}^2) \\ (m_{21}^1, m_{21}^2) \end{array} \right. & \left. \begin{array}{c} (m_{12}^1, m_{12}^2) \\ (m_{22}^1, m_{22}^2) \end{array} \right) \end{array},$$

where

$$m_{11}^1 = p_1 x + v_N^1(p_1) + p_2 v_N^1(p_1) + (1 - p_1 - p_2) v_N^{2,1};$$
$$m_{11}^2 = p_2 x + p_1 v_N^1(p_2) + (1 - p_1 - p_2) v_N^{2,2};$$
$$m_{12}^1 = p_1 x + v_N^{2,1} + (1 - p_1) v_N^{2,1};$$
$$m_{12}^2 = p_1 v_N^1(p_2) + \bar{p}_1 v_N^{2,2};$$
$$m_{21}^1 = p_2 v_N^1(p_1) + \bar{p}_2 v_N^{2,1};$$
$$m_{21}^2 = p_2 x + \bar{p}_2 v_N^{2,2};$$
$$m_{22}^1 = v_N^{2,1};$$
$$m_{22}^2 = v_N^{2,2}.$$

$$v_i^{2,j} = \int_0^{v_{i+1}^{2,j}} v_{i+1}^1 \, dx + \int_{v_{i+1}^{2,j}}^1 (p_j x + \bar{p}_j v_{i+1}^{2,j}) \, dx = v_i^1(p_j); j = 1, 2.$$

m players

The expected payoff of the j-th player at the stage i is $v_i^{m,j}, j = 1, 2, ..., m, i = 1, ..., N.$

The player j accepts the i-th applicant with quality value x if

$x \geq v_{i+1}^{m,j},\ i = 1, 2, ..., N - 1.$

Theorem 1 *In the m-person best-choice game each player uses an optimal strategy as if the other players were not there, that is, $v_i^{m,j} = v_i^1(p_j), j = 1, 2, ..., m; i = 1, ..., N - 1; v_N^1(p_j) = \frac{p_j}{2} + \bar{p}_j C$ for every m.*

m-person best-choice game with two stops

- Each of m companies (players) wants to employ two secretaries among N applicants.
- Each player observes the value of applicant's quality and decides either to accept or to reject the applicant.
- Applicants' qualities have uniform distribution on [0,1].
- If player j accepts an applicant then there is probability p_j that the applicant rejects the proposal $j = 1, 2, ..., m$.
- If player j employs two secretaries then he leaves the game. The payoff of the player is equal to sum of the expected quality values of selected secretaries.
- Applicant rejected by player cannot be considered later.
- The shortfall of a player not employing any applicant is C, $C \in [0, 1]$.
- Each player aims to maximize his expected payoff.

One player

$v_i^1(p_j)$ — expected payoff of the player at the stage i

$v_{i,r}^1(p_j)$ — expected payoff of the player at the stage r on condition he has already employed a secretary at the stage i

The expected player's payoff if he stays in the game alone is following

$$
\begin{aligned}
&v_i^1(p_j) = \mathbf{E}\left(\max\left\{p_j(X_i + v_{i,i+1}^1(p_j)) + \bar{p}_j v_{i+1}^1(p_j); v_{i+1}^1(p_j)\right\}\right), i = 1, 2, ..., N,\\
&v_{N+1}^1(p_j) = -C;\\
&v_{i,r}^1(p_j) = \mathbf{E}\left(\max\left\{p_j X_r + \bar{p}_j v_{i,r+1}^1(p_j); v_{i,r+1}^1(p_j)\right\}\right), r = i+1, ..., N,\\
&v_{i,N+1}^1(p_j) = -C.
\end{aligned}
$$

If the player has already employed an applicant at the stage i, he accepts another applicant if $x \geq v_{i,r+1}^1(p_j)$.

The first applicant would be accepted at the stage i if $x \geq v_{i+1}^1(p_j) - v_{i,i+1}^1(p_j)$.

$$v_i^1 = v_{i,i+1}^1 + \int\limits_0^{v_{i+1}^1 - v_{i,i+1}^1} (v_{i+1}^1 - v_{i,i+1}^1)dx + \int\limits_{v_{i+1}^1 - v_{i,i+1}^1}^{1} (p_j x + \bar{p}_j (v_{i+1}^1 - v_{i,i+1}^1))dx$$
$$= v_{i+1}^1 + \frac{p_j}{2}(1 - (v_{i+1}^1 - v_{i,i+1}^1))^2;$$
$$v_{i,r}^1 = \int\limits_0^{v_{i,r+1}^1} v_{i,r+1}^1 dx + \int\limits_{v_{i,r+1}^1}^{1} (p_j x + (1 - p_j) v_{i,r+1}^1)dx = v_{i,r+1}^1 + \frac{p_j}{2}(1 - v_{i,r+1}^1)^2;$$
$$v_{i,N}^1 = \frac{p_j}{2} - \bar{p}_j C;$$
$$v_{i,r}^1 = v_{i,r}^1(p_j); v_i^1 = v_i^1(p_j), i = 1, ..., N-1, r = i+1, ..., N.$$

Table 2. Optimal thresholds for $N = 10$, $p_j = 0$, $C = 0$

i	1	2	3	4	5	6	7	8	9	10
$v_{i+1}^1 - v_{i,i+1}^1$	0.757	0.735	0.708	0.676	0.634	0.579	0.5	0.375	0	0
$v_{i,i+1}^1$	0.850	0.836	0.820	0.800	0.775	0.742	0.695	0.625	0.5	0

Two players

$v_i^{2,j}$ — expected payoff of the j-th player at the stage i

$v_{i,r}^{2,j}, j = 1, 2$ — expected payoff of the j-th player at the stage r on condition he has already employed a secretary at the stage i

At the stage $N-2$ if the first player hasn't employed a secretary and the second player selected one, the matrix of the game is as following:

$$M_{N-2}^2(x) = \begin{array}{c} \\ \mathsf{A}_1 \\ \mathsf{R}_1 \end{array} \begin{array}{cc} \mathsf{A}_2 & \mathsf{R}_2 \\ \left(\begin{array}{cc} (m_{11}^1, m_{11}^2) & (m_{12}^1, m_{12}^2) \\ (m_{21}^1, m_{21}^2) & (m_{22}^1, m_{22}^2) \end{array} \right) \end{array},$$

where

$$m_{11}^1 = p_1(x + v_{N-2,N-1}^{2,1}) + p_2 v_{N-1}^1(p_1) + (1 - p_1 - p_2) v_{N-1}^{2,1};$$
$$m_{11}^2 = p_2 x + p_1 v_{i,N-1}^{2,2} + (1 - p_1 - p_2) v_{i,N-1}^{2,2};$$
$$m_{12}^1 = p_1(x + v_{N-2,N-1}^{2,1}) + (1 - p_1) v_{N-1}^{2,1};$$
$$m_{12}^2 = p_1 v_{i,N-1}^1(p_2) + \bar{p}_1 v_{i,N-1}^{2,2};$$
$$m_{21}^1 = p_2 v_{N-1}^1(p_1) + \bar{p}_2 v_{N-1}^{2,1};$$
$$m_{21}^2 = p_2 x + \bar{p}_2 v_{i,N-1}^{2,2};$$
$$m_{22}^1 = v_{N-1}^{2,1};$$
$$m_{22}^2 = v_{i,N-1}^{2,2}.$$

m-person game

$v_i^{m,j}, j = 1, 2, ..., m$ — expected payoff of the j-th player at the stage i

$v_{i,r}^{m,j}, j = 1, 2, ..., m$ — expected payoff of the j-th player at the stage r on condition he has already employed a secretary at the stage i

Theorem 2 *in the m-person best-choice game each player uses an optimal strategy as if the other players were not there, that is,* $v_i^{m,j} = v_i^1(p_j), i = 1, ..., N-1;$ $v_{i,r}^{m,j} = v_{i,r}^1(p_j), r = i+1, ..., N; v_{i,N}^1(p_j) = \frac{p_j}{2} + \bar{p}_j C, j = 1, 2, ..., m.$

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