Central Limit Theorem for Analitic Functions of two-sided moving averages. Limit theorem for canonical U-statistics

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2nd Northern Triangular Seminar, 2010

Let $\{\xi_j\}_{j\in\mathbb{Z}}$ be a stationary sequence, and $\{a_j;\, j\in\mathbb{Z}\}$ be real numbers.

Moving-average sequence (*linear process*) is defined by

$$X_k := \sum_{j\in\mathbb{Z}} a_{k-j}\xi_j. \tag{1}$$

If

$$\mathbf{E}|\xi_0| < \infty \quad \text{and} \quad \sum_{j\in\mathbb{Z}} |a_j| < \infty \quad \text{or} \tag{2}$$

$$\{\xi_j\}_{j\in\mathbb{Z}} \text{ are i.i.d.,} \quad \mathbf{E}\xi_0 = 0, \quad \mathbf{E}\xi_0^2 < \infty, \quad \text{and} \quad \sum_{j\in\mathbb{Z}} a_j^2 < \infty, \tag{3}$$

then (1) is well defined.

Outline

1. Mixing conditions

2. CLT for the sequence $\{g(X_k)\}_{k\geq 1}$, where $g(x)$ is a non-linear function.

3. Limit theorem for canonical U-statistics.

Mixing conditions

[Rosenblatt, 1956] A stationary sequence $\{X_k\}_{k\in\mathbb{Z}}$ is called *strong mixing*, or *a-mixing*, if $\alpha(m) \to 0$ as $m \to \infty$ where

$$\alpha(m) = \sup_{A\in\mathcal{F}_{-\infty}^{0},\ B\in\mathcal{F}_{m}^{\infty}} |\mathbf{P}(B\cap A) - \mathbf{P}(B)\,\mathbf{P}(A)|, \tag{4}$$

[Ibragimov, 1962] *uniformly strong mixing* or φ-mixing: $\varphi(m) \to 0$ as $m \to \infty$ where

$$\varphi(m) = \sup_{\substack{A\in\mathcal{F}_{-\infty}^{0},\ B\in\mathcal{F}_{m}^{\infty},\\ \mathbf{P}(A)\neq 0}} \frac{|\mathbf{P}(B\cap A) - \mathbf{P}(B)\,\mathbf{P}(A)|}{\mathbf{P}(A)} \tag{5}$$

where $\mathcal{F}_{-\infty}^{0} = \sigma\{X_j, j \le 0\}$ and $\mathcal{F}_{m}^{\infty} = \sigma\{X_j, j \ge m\}$.

[Ibragimov, Linnik, 1965] If $\{\xi_j\}$ is i.i.d Gaussian sequence then φ-mixing is equivalent to the finitness of $A_0 := \{j \in \mathbb{Z} : a_j \neq 0\}$.

[Rosenblatt 1980, Andrews 1984] Let ξ_j be independent Bernoulli random variables $|\rho| < 1$. Then the sequence $X_k = \sum_{j=0}^{\infty} \rho^j \xi_{k-j}$ is not α-mixing.

Theorem 1. Let $\{\xi_j;\ j \in \mathbb{Z}\}$ be nondegenerated independent descrete random variables. Moreover, let $0 < \delta \leq \Delta < \infty$ be such constants that distance between any two atoms of ξ_j is not less than δ and is not greater than Δ. Finally, let the series (2) be convergent for each k and one of the following two conditions be fulfilled

1) $a_j = a_0 q^{|j|}$ if $j < 0$ and $a_j = a_0 a^j$ if $j \geq 0$, where $a_0 \neq 0$, $0 < q < 1$, $0 < a \leq \delta/(\Delta + \delta)$;

2) $0 < |a_{k_{j+1}}| \leq |a_{k_j}|\delta/(\Delta + \delta)$ for all $j \geq 0$, where $\{a_{k_j};\ j \geq 0\}$ are non-zero coefficients a_j ordered by their absolute values.

Then the the sequence $\{X_k\}$ does not satisfy α-mixing.

If there are only strong inequalities for a and a_{k_j} in conditions 1) and 2) then proposition is true when ξ_j are arbitrary dependent.

Theorem 2. Let $\{\xi_j;\ j \in \mathbb{Z}\}$ be independent bounded random variables. Let set

$$A^{-} := \{j < 0 : a_j \neq 0\}$$

be infinite and $\sum_{j\in\mathbb{Z}} |a_j| < \infty$. Then the sequence $\{X_k\}$ does not satisfy φ-mixing condition.

Theorem 3. Let $\{\xi_j;\ j \in \mathbb{Z}\}$ be independent bounded random variables with density $p(x)$, A^{-} be finite and for some $C > 0$

$$\int_{\mathbb{R}} |p(y+x) - p(y)|\, dy \leq C|x| \quad \text{for all } x \in \mathbb{R}.$$

$$\sum_{j\in\mathbb{Z}} j|a_j| < \infty,$$

$$|a_{k_0}| > \sum_{j\neq k_0} |a_j| \quad \text{for some } k_0.$$

Then $\{X_k\}$ satisfies φ-mixing condition.

Theorem 4. Let $\{\xi_j;\ j \in \mathbb{Z}\}$ be independent nondegenerated random variables, $a_j > 0$ for all j, and the following conditions be fulfilled

1) for some positive constants x_0, c_0, c_1, c_2, integers $j_1 > 0$ and j_0

$$\begin{aligned} &\sup_{x\geq x_0} \tfrac{\mathbf{P}(\xi_{j_0}\geq x+y)}{\mathbf{P}(\xi_{j_0}\geq x)} \leq c_1 e^{-c_2 y} && \text{for all } y \geq 0, \\ &\mathbf{P}(\xi_j \geq x) \leq c_1 e^{-c_2 x} && \text{for all } x \geq x_0, \text{ if } |j| < j_1, \\ &\mathbf{P}(\xi_j \geq x) \leq c_0 \mathbf{P}(\xi_{j_0} \geq x) && \text{for all } x \geq x_0, \text{ if } |j| \geq j_1; \end{aligned}$$

2) inequality holds

$$\inf_{j\in\mathbb{Z}} \frac{a_j}{a_{j+1}} > 0;$$

and one of the following conditions is fulfilled

3) inequalities hold

$$\sum_{j\neq 0} a_j \ln|j| < \infty \quad \text{and} \quad \inf_{j\in\mathbb{Z}} \frac{a_{j+1}}{a_j} > 0 \quad \text{or}$$

3′) for some $\delta > 0$, $c_3 > 0$

$$\sum_{j\in\mathbb{Z}} a_j |j|^\delta < \infty \quad \text{and} \quad \frac{a_{j+1}}{a_j} \ln|j| \geq c_3 \quad \text{for all } j \text{ such that } |j| \text{ is greate enough.}$$

Then the corresponding sequence $\{X_k\}$ does not satisfy φ-mixing condition.

CLT for functions of moving averages

Results for $g(X_k) = h(\{\xi_{k-j}\}_{j\in\mathbb{Z}})$. Let $g(x)$ be a Lipschitz function and $\{\xi_j\}$ be i.i.d.

Ibragimov, Linnik, 1965, Billingsley, 1968

$$\sum_{n=1}^{\infty}\Big(\sum_{|k|\geq n} a_k^2\Big)^{1/2} < \infty \tag{10}$$

Ibragimov, Linnik, 1965

$$\mathbf{E}|X_1|^{2+\delta} < \infty, \quad \sum_{n=1}^{\infty}\Big(\mathbf{E}\Big|\sum_{|k|\geq n} a_k\xi_{-k}\Big|^{\frac{2+\delta}{1+\delta}}\Big)^{\frac{1+\delta}{2+\delta}} < \infty, \tag{11}$$

or

$$|X_1| < C, \quad \sum_{n=1}^{\infty}\mathbf{E}\Big|\sum_{|k|\geq n} a_k\xi_{-k}\Big| < \infty. \tag{12}$$

Hall, Heyde, 1975.

$$\sum_{n=1}^{\infty}\Big(\sum_{|k|\ge n}^{\infty}|a_k|\Big)^2<\infty, \tag{13}$$

Conditions (10)–(13) imply

$$\sum_{j\in\mathbb{Z}}|a_j|<\infty. \tag{14}$$

Conditions (10), (13) are stronger than (14). Also if ξ_j are Gaussian then conditions (11), (12) imply (10).

Ho, Hsing (1997). $a_j=0$ for all $j<0$,

$$\sum_{j\in\mathbb{Z}}|a_j|<\infty.$$

Wu (2002). $a_j=0$ for all $j<0$,

$$\sum_{n=1}^{\infty}\Big(\sum_{k=n}^{\infty}a_k\Big)^2<\infty.$$

[Dedecker J., Merlevede F., Volny D. (2007)]. If $\sum_{j\in\mathbb{Z}}|a_j| < \infty$, $\{\xi_j\}$ – i.i.d, $\mathbf{E}\xi_0 = 0$, $\mathbf{E}\xi_0^2 < \infty$,

$$w_g(h, M) \le ChM^{\alpha},$$

where $\alpha \ge 0$, $\mathbf{E}|\xi_0|^{2+2\alpha} < \infty$, or

$$\|\xi_0\| < \infty,$$

$$\sum_{k\in\mathbb{Z}} w_g(c|a_k|, \|X_0\|_\infty) < \infty,$$

where

$$w_g(h, M) = \sup_{|t|\le h, |x|\le M, |x+t|\le M} |g(x+t) - g(x)|.$$

Then CLT holds.

[Dedecker J., Merlevede F., Volny D. (2007)]. If $a_k = 0$, $k < 0$, $\{\xi_j\}$ – i.i.d, $\mathbf{E}\xi_0 = 0$, $\mathbf{E}\xi_0^2 < \infty$,

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^n |a_i|}{|\sum_{i=0}^n a_i|} < \infty, \quad \text{and} \quad \sum_{k=1}^n \sqrt{\sum_{i\ge k} a_i^2} = o(s_n),$$

where $s_n = \sqrt{n}|a_0 + \ldots + a_n|$, g is Lipschitz and g' is contnuous then

$$\frac{1}{s_n}\sum_{k=1}^n g(X_k) \to_d \mathcal{N}(0,\sigma^2), \quad \text{where} \quad \sigma^2 = \mathbf{E}\xi_0^2\Big(\mathbf{E}g'(X_0)\Big)^2.$$

[Dedecker J., Merlevede F., Volny D. (2007)]. If $a_k = 0$, $k < 0$, $\{\xi_j\}$ – i.i.d, $\mathbf{E}\xi_0 = 0$, $\mathbf{E}\xi_0^4 < \infty$,

$$\sum_{k\ge 0} |a_k| \sqrt{\sum_{i=k+1}^{\infty} a_i^2} < \infty, \quad \mathbf{E}g'(X_0) = 0.$$

g' is Lipschitz, then CLT holds.

Theorem 5. Let $g(X_k) := \sum_{l \geq 0} \beta_l X_k^l$, $C = \sum |a_i|$ and one of the following conditions (α) or (α_0) be fulfilled:

(α) for some $\delta > 0$

$$\begin{cases} \sum\limits_{l \geq 0} |\beta_l| \cdot l \cdot C^l \Big(\mathbf{E} |\xi_0|^{(2+\delta)l} \Big)^{\frac{1}{2+\delta}} < \infty, \\ \sum\limits_{n=0}^{\infty} \alpha(n)^{\frac{\delta}{2+\delta}} < \infty, \quad \text{or} \end{cases}$$

(α_0) ξ_0 is bounded and

$$\begin{cases} \sum\limits_{l \geq 0} |\beta_l| \cdot l \cdot C^l < \infty \\ \sum\limits_{n=0}^{\infty} \alpha(n) < \infty, \end{cases}$$

where $\alpha(n)$ is a mixing coefficient for $\{\xi_j\}$.

Then $\{g(X_k)\}_{k \geq 1}$ satisfies CLT.

Theorem 6. Let $\{\xi_j\}$ be i.i.d, $\beta_l = 0$ and $\mathbf{E}\xi_0^l = 0$ for all odd numbers l,

$$\sum_{k \geqslant 0} \Big(\sum_i |a_i a_{i+k}| \Big)^2 < \infty,$$

$$\sum_{l \geqslant 0} |\beta_l| \Big(2 \sum_i a_i^2 \Big)^{l/2} (l!)^{1/2} \left(\mathbf{E}|\xi|^{2l} \right)^{1/2} < \infty.$$

Then $\{g(X_k)\}_{k \geq 1}$ satisfies CLT.

Limit theorem for U-statistics

Second order degenerated (canonical) U-statistics:

$$U_n = \frac{1}{n} \sum_{1 \le k_1 \ne k_2 \le n} f(X_{k_1}, X_{k_2}) \tag{40}$$

where kernel f is degenerated (canonical), i.e.

$$\mathbf{E} f(t, X_k) = \mathbf{E} f(X_k, t) = 0 \quad \text{for all} \quad t. \tag{41}$$

Rubin H., Vitale R. A. (1980). $\{X_k\}_{k \ge 1}$ are independent.
Borisov I. S., Volodko N. V. (2008). $\{X_k\}_{k \ge 1}$ are m-dependent or mixing.

Let functions $\{e_i(t) : i \geq 0\}$ form an orthonormal basis in

$$\{h : \mathbf{E}h^2(X_0) < \infty\},$$

$e_0(t) \equiv 1$.

Then $\mathbf{E}e_i(X_0) = 0$ for all $i \geq 1$, $\mathbf{E}e_i^2(X_0) = 1$ for all i.

Functions $\{e_i(t_1)e_j(t_2)\}_{i,j\geq 0}$ form an orthonormal basis in

$$\{h : \mathbf{E}h^2(X_0^*, X_1^*) < \infty\}.$$

Let $\mathbf{E}f^2(X_0^*, X_1^*) < \infty$, then

$$f(t_1, t_2) = \sum_{i_1, i_2 \geq 1} f_{i_1,i_2} e_{i_1}(t_1) e_{i_2}(t_2). \tag{43}$$

Theorem 7. Let function f be continuos, functions e_i and e_ie_j be Lipschitz for all $i,j \geq 1$,

$$\sum_i |a_i| < \infty, \tag{44}$$

$$\sum_{i,j\geq 1} |f_{i,j}|(1 + Lip(e_i)Lip(e_j)) < \infty. \tag{45}$$

Then

$$U_n \xrightarrow{d} \sum_{i_1,i_2\geq 1} f_{i_1,i_2} H_{i_1,i_2}(\tau_{i_1}, \tau_{i_2}), \tag{46}$$

where $\{\tau_j\}$ is a Gaussian sequence of centered random variables with covariations

$$\sigma_{i,j} = cov(\tau_i, \tau_j) = \sum_{k=-\infty}^{\infty} cov(e_i(X_0), e_j(X_k)), \tag{47}$$

$H_{i_1,i_2}(\tau_{i_1}, \tau_{i_2}) = \tau_{i_1}\tau_{i_2}$ for $i_1 \neq i_2$, and $H_{i,i}(\tau_i, \tau_i) = \tau_i^2 - 1$.

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