Nelson-type Limit for a Particular Class of Lévy Processes

Haidar Al-Talibi

School of Computer Science, Physics and Mathematics
Linnaeus University

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Outline

In the physical model $x(t)$ describes the position of a Brownian particle at time $t > 0$. It is assumed that the velocity $\frac{dx}{dt} = v$ exists and satisfies the Langevin equation.

Mathematically the two ordinary differential equations combine to the initial value problem:

$$\begin{aligned} dx_t &= v_t\, dt \\ dv_t &= -\beta v_t dt + \beta K(x_t) dt + dB_t, \end{aligned} \tag{1}$$

with initial value $(x_0, v_0) = (x(0), v(0))$, where $B_t,\ t \geq 0$, is mathematical Brownian motion on the real line and $\beta > 0$ is a constant which physically represents the inverse relaxation time between two successive collisions. $K(x,t)$ is an external field of force.

Moreover sufficient conditions for the existence of a unique solution of (1) can be found in e.g. [Applebaum] and in [Kolokoltsov, Schilling, Tyukov].

For the physical Ornstein Uhlenbeck theory of motion, given by a second order SDE on $\boldsymbol{R}^d$, the solution of the corresponding system on the cotangent bundle ($\boldsymbol{R}^2$) is given by:

$$v_t = e^{-\beta t} v_0 + \beta \int_0^t e^{-\beta(t-u)} K(x_u) du + \int_0^t e^{-\beta(t-u)} dB_u,$$

which is called Ornstein-Uhlenbeck velocity process, and

$$x_t = x_0 + \int_0^t e^{-\beta s} v_0 ds + \beta \int_0^t \int_0^s e^{-\beta(s-u)} K(x_u) duds + \int_0^t \int_0^s e^{-\beta s} e^{\beta u} dB_u ds, \quad (2)$$

which is called Ornstein-Uhlenbeck position process. The initial values are given by $(x_0, v_0) = (x(0), v(0))$ and $t \geq 0$.

Remark We introduce this physical notation for the Ornstein-Uhlenbeck process since it is more adequate for our studies than the mathematical one. In Nelson's notation the noise B is Gaussian with variance $2\beta^2 D$ with $2\beta^2 D = 2\frac{\beta k T}{m}$ and physical constants k, T, m in order to match Smolouchwsky's constants.

Let us modify the Ornstein-Uhlenbeck process (2) as in [Al-Talibi, Hilbert, Kolokoltsov].
We introduce a stochastic Newton equation driven by βX_t, where $\{X_t\}_{t\geq 0}$ is an α-stable Lévy process, with $0 < \alpha < 2$ and β is a scaling parameter.
Sufficient conditions for the existence of a unique solution may be found in [Applebaum] and [Kolokoltsov, Schilling, Tyukov]. In this case the solution of this stochastic differential equation can be represented as given in the proposition below.

Proposition

Assume $A : \boldsymbol{R} \to \boldsymbol{R}$ *is linear. Furthermore, let* X *be a Lévy process on* $\boldsymbol{R}$*. Let* $f : [0, \infty] \to \boldsymbol{R}$ *be a continuous function. Then the solution of the stochastic differential equation*

$$dx_t = Ax_t dt + f(t)dt + dX_t, \qquad t \geq 0$$

with initial value $x(0) = x_0$*, is*

$$x_t = e^{At}x_0 + \int_0^t e^{A(t-s)} f(s)ds + \int_0^t e^{A(t-s)} dX_s.$$

Proof.

The equation in question has a unique solution, [Applebaum]. We can verify the solution using integration by parts respectively Itô formula, i.e.

$$e^{-At}x_t = x_0 + \int_0^t x_s\left(-Ae^{-As}\right)ds + \int_0^t e^{-As}dx_s,$$

and inserting for $dx_t = Ax_t dt + f(t)dt + dX_t$ we obtain

$$e^{-At}x_t = x_0 + \int_0^t e^{-As}f(s)ds + \int_0^t e^{-As}dX_s,$$

and we are done. □

Let $(x,\ v)$ be the solution of the system

$$\begin{aligned} dx(t) &= v(t)dt, \qquad x(0) = x_0, \\ dv(t) &= -\beta v(t)dt + \beta K(x(t),t)dt + dB_t, \qquad v(0) = v_0\ . \end{aligned}$$

where the noise B is Gaussian with variance β^2 on $\boldsymbol{R}^\ell$.

Theorem 10.1 Let $(x,\ v)$ satisfy the equation above and assume that K is a function in $\boldsymbol{R}^\ell$ satisfying a global Lipschitz condition. Moreover assume that B is standard BM and y solves the equation

$$dy(t) = K(y(t),t)dt + dB(t) \qquad y(0) = v_0\ .$$

Then for all v_0 with probability one

$$\lim_{\beta\to\infty} x(t) = y(t),$$

uniformly for t in compact subintervals of $[0,\infty)$.

Here we introduce a modified Ornstein-Uhlenbeck position process driven by βX_t, where $\{X_t\}_{t\geq 0}$ is an α-stable Lévy process, $0 < \alpha < 2$ and $\beta > 0$ is a scaling parameter as above. Let us focus on the position process $\{x_t\}_{t\geq 0}$. Due to Proposition 1, the solution has the form

$$x_t = x_0 + \int_0^t e^{-\beta s} v_0 ds + \beta \int_0^t \int_0^s e^{-\beta(s-u)} K(x_u) duds + \int_0^t \int_0^s \beta e^{-\beta s} e^{\beta u} dX_u ds, \quad (3)$$

where K satisfies sufficient conditions to guarantee existenc and uniqueness of solutions see e.g. [Applebaum] and [Kolokoltsov, Schilling, Tyukov].

For arbitrary Lévy processes Y the characteristic function is of the form $\phi_{Y_t}(u) = e^{t\eta(u)}$ for each $u \in \boldsymbol{R},\ t \geq 0$, η being the Lévy-symbol of $Y(1)$.

We concentrate on α-stable Lévy processes with Lévy-symbol for $\alpha \neq 1$:

$$\eta(u) = -\sigma^{\alpha}|u|^{\alpha}\left[1 - i\gamma\,\mathrm{sgn}\,(u)\tan\left(\frac{\pi\alpha}{2}\right)\right] \tag{4a}$$

and for $\alpha = 1$ is:

$$\eta_1(u) = -\sigma|u|\left[1 + i\gamma\frac{2}{\pi}\mathrm{sgn}\,(u)\log\left(|u|\right)\right] \tag{4b}$$

for constant γ.

Proposition (Lukacs)

Assume that Y is an α-stable Lévy process, $0 < \alpha < 2$, and g is a continuous function on the interval $[s,t] \subset T \subsetneq \boldsymbol{R}$.

Let η be the Lévy symbol of Y_1 and
ξ be the Lévy symbol of $\psi(t) = \int_s^t g(r)\, dY_r$.

Then we have

$$\xi(u) = \int_s^t \eta(ug(r))\, dr \, .$$

For $g(\ell) = e^{\beta(\ell - t)}, \ell \geq 0$ and the α-stable process X in (3) the symbol of $Z_t = \int_s^t e^{\beta(r-t)}\, dX_r$ is:

$$\xi(u) = \begin{cases} \int_s^t e^{\alpha\beta(r-t)}\, dr \cdot \eta(u) & \text{for } 0 < \alpha < 2, \alpha \neq 1 \\ \int_s^t e^{\alpha\beta(r-t)}\, dr \cdot \eta_1(u) & \text{for } \alpha = 1 \end{cases}$$

with η, η_1 as in (4a) and (4b), respectively, and $0 \leq s \leq t$.

For $g(\ell) = e^{\beta(\ell - t)}, \ell \geq 0$ and the α-stable process X in (3) the symbol of $Z_t = \int_s^t e^{\beta(r-t)}\, dX_r$ is:

$$\xi(u) = \begin{cases} \int_s^t e^{\alpha\beta(r-t)}\, dr \cdot \eta(u) & \text{for } 0 < \alpha < 2, \alpha \neq 1 \\ \int_s^t e^{\alpha\beta(r-t)}\, dr \cdot \eta_1(u) & \text{for } \alpha = 1 \end{cases}$$

with η, η_1 as in (4a) and (4b), respectively, and $0 \leq s \leq t$.

We are thus lead to introduce the time change $\tau^{-1}(t)$ where

$$\tau(t) = \int_0^t e^{-\alpha\beta t} e^{\alpha\beta u} du = \frac{1}{\alpha\beta}\left(1 - e^{-\alpha\beta t}\right) \tag{5}$$

which is actually deterministic.

This means that X_t and $Z_{\tau^{-1}(t)}$ have the same distribution.

Theorem

Let $t_1 < t_2$, $t_1, t_2 \in T$, and T a compact subset of $[0, \infty)$. Then there are N_1 and N_2 satisfying:

$$(i)\ t_2 - t_1 \geq \frac{N_1}{\beta} \quad and \quad (ii)\ \beta^{\alpha} \geq N_2 v_0^{\alpha}, \tag{6}$$

with $0 < \alpha < 2$. Furthermore let

$$dy_t = K(y_t)dt + dX_t, \tag{7}$$

with $y(0) = x_0$ and $K : R^d \to R^d$ satisfy a global Lipschitz condition, then

$$\lim_{\beta \to \infty} x_t = y_t,$$

for any $t \in T$ where $\{x_t\}_{t \geq 0}$ is the Ornstein-Uhlenbeck position process (3) and $\{y_t\}_{t \geq 0}$ is the solution of (7) with $\{X_t\}_{t \geq 0}$ as its driving α-stable Lévy noise.

For the increment of the OU position process

$$x_{t_2} - x_{t_1} = \int_{t_1}^{t_2} e^{-\beta s} v_0 ds + \beta \int_{t_1}^{t_2} \int_0^s e^{-\beta(s-u)} K(x_u) du ds + \int_{t_1}^{t_2} \int_0^s e^{-\beta(s-u)} \beta dX_u ds. \tag{8}$$

The first integral of (8) is $\int_{t_1}^{t_2} e^{-\beta s} v_0 ds = \frac{v_0}{\beta}\left(e^{-\beta t_1} - e^{-\beta t_2}\right)$.
Taking the latter expression to the power α, where $0 < \alpha < 2$ and taking into account that $e^{-\beta t_1} - e^{-\beta t_2} \leq 1$ we obtain that

$$\frac{v_0^\alpha}{\beta^\alpha} \left|e^{-\beta t_1} - e^{-\beta t_2}\right|^\alpha \leq \frac{1}{N_2} e^{-\alpha N_1} \left|-(1 - e^{-N_1})\right|^\alpha .$$

Where we used $((6)(i), (ii))$ and the fact that $e^{-\alpha\beta t_1} \leq e^{-\alpha\beta\Delta t} \leq e^{-\alpha N_1}$. If we choose N_1 and N_2 large enough then $\frac{1}{N_2} e^{-\alpha N_1} \left|-(1 - e^{-N_1})\right|^\alpha$ tends to zero.

The third part of (8) is estimated by first splitting the double integral into two integrals. We have

$$\beta\left[\int_{t_1}^{t_2}\int_{t_1}^{s} e^{-\beta s}e^{\beta u}dX_u ds + \int_{t_1}^{t_2}\int_{0}^{t_1} e^{-\beta s}e^{\beta u}dX_u ds\right]. \quad (9)$$

The double integral of the second part of (9) tends to zero as β and N_1 tend to infinity,

$$\beta\int_{t_1}^{t_2}\int_{0}^{t_1} e^{-\beta s}e^{\beta u}\,dX_u ds = -Z_{\tau(t_1)}\left(e^{-\beta t_2} - e^{-\beta t_1}\right)e^{\beta t_1}$$

$$= \left(1 - e^{-\beta\Delta t}\right)Z_{\frac{1}{\alpha\beta}\left(1-e^{-\alpha\beta t_1}\right)} = \frac{1}{\sqrt[\alpha]{\beta}}\left(1 - e^{-\beta\Delta t}\right)Z_{\frac{1}{\alpha}\left(1-e^{-\alpha\beta t_1}\right)},$$

where Z_τ is an α-stable Lévy process. Moreover, the scaling property of Lévy processes we used in the last step, i.e. $Z_{\gamma\tau} = \gamma^\alpha Z_\tau$, where $\gamma > 0$, is actually a special case of Proposition 2.

Using the assumption (6(*i*)) we obtain

$$e^{-\beta \Delta t} \leq e^{-N_1}$$

Thus, for large N_1 and large β, the latter expression converges to zero and $Z_{\frac{1}{\alpha}\left(1-e^{-\alpha\beta t_1}\right)}$ converges to $Z_{\frac{1}{\alpha}}$ a.e. which is almost surely finite. Hence the product converges almost surely to zero.

Let us turn to the first part of (9) which reveals the increment of the driving Lévy process. We use partial integration to have

$$\beta \int_{t_1}^{t_2} \int_{t_1}^{s} e^{-\beta s} e^{\beta u} dX_u ds = -e^{-\beta t_2} \int_{t_1}^{t_2} e^{\beta u} dX_u + \left(X_{t_2} - X_{t_1}\right). \tag{10}$$

By introducing a time change in analogy to (5) on the right hand side of (10) we obtain

$$-e^{-\beta t_2}\int_{t_1}^{t_2} e^{\beta u} dX_u = Z_{\frac{1}{\alpha\beta}(1-e^{-\alpha\beta\Delta t})} = \frac{1}{\sqrt[\alpha]{\beta}} Z_{\frac{1}{\alpha}(1-e^{-\alpha\beta\Delta t})},$$

where we used the scaling property of Lévy processes $Z_{\gamma\tau} = \gamma^{\alpha} Z_{\tau}$ with $\gamma > 0$. By assumption $(6(i))$ we see that $e^{-\alpha\beta\Delta t} \le e^{-\alpha N_1}$ which tends to zero for large N_1 and $Z_{\frac{1}{\alpha}(1-e^{-\alpha\beta\Delta t})}$ converges to $Z_{\frac{1}{\alpha}}$. In analogy to the argument above the product $\frac{1}{\sqrt[\alpha]{\beta}} Z_{\frac{1}{\alpha}\left(1-e^{-N_1}\right)}$ tends to zero almost surely for N_1 and β tending to infinity.

The second part of (8) can be rewritten as (by using integration by parts)

$$\left[-e^{-\beta s}\int_0^s e^{\beta u}K(x_u)\right]_{t_1}^{t_2}+\int_{t_1}^{t_2}K(x_s)ds=$$
$$=e^{-\beta t_1}\int_0^{t_1}e^{\beta u}K(x_u)du-e^{-\beta t_2}\int_0^{t_2}e^{\beta u}K(x_u)du+\int_{t_1}^{t_2}K(x_s)ds. \tag{11}$$

The first integral of (11) can be written as

$$\left|\int_0^{t_1}e^{-\beta(t_1-u)}K(x_u)du\right|\leq\int_0^{t_1}e^{-\beta(t_1-u)}\left|K(x_u)-K(x_0)\right|du+K(x_0)\int_0^{t_1}e^{-\beta(t_1-u)}du. \tag{12}$$

Let κ be the Lipschitz constant of K such that $|K(x_1) - K(x_2)| \leq \kappa |x_1 - x_2|$ for $x_1, x_2 \in \boldsymbol{R}$.
Looking at the first integral in (12) we see that it is bounded by

$$\int_0^{t_1} e^{-\beta(t_1 - u)} \, |K(x_u) - K(x_0)| \, du \leq t_1 \kappa \sup_{0 \leq u \leq t_1} |x_u - x_0| \int_0^{t_1} e^{-\beta(t_1-u)} du$$

Now consider (3), observing that $\beta \int_0^s e^{-\beta(s-u)} du \leq 1$ we can write

$$|x_t - x_0| \leq |v_0| + \int_0^t K(x_u) du + \int_0^t dX_u \tag{13}$$

The second integral of (13) is bounded in absolute value by $tK(x_0) + t\kappa |x_u - x_0|$.
Letting $t\kappa \leq \frac{1}{2}$ and taking the supremum of (13) for all $0 \leq t \leq t_1$ we obtain

$$\sup_{0\le t\le t_1}|x_t - x_0| \le |v_0| + \frac{1}{2}\sup_{0\le u\le t_1}|x_u - x_0| + \frac{1}{2\kappa}|K(x_0)| + \sup_{0\le u\le t_1}|X_u - X_0|$$

Rearranging we obtain that

$$\sup_{0\le t\le t_1}|x_t - x_0| \le 2|v_0| + \frac{1}{\kappa}|K(x_0)| + 2\sup_{0\le u\le t_1}|X_u - X_0|.$$

Convergence in probability ?

Restrict $1 < \alpha < 2$?

Using induction

$$\zeta_n = \sup_{t_n\le t\le t_{n+1}}\left|x_t - x_{t_n}\right|.$$

is bounded for all $t_n \le t \le t_{n+1}$, $n = 0, 1, 2, \cdots$, and any $t \in [0, T]$.
The second integral in (11) is treated in the same manner.

What is left? Finally, the remaining part of (11) is the integral $\int_{t_1}^{t_2} K(x_s)ds$ and the increments of α-stable Lévy process $X_{t_2} - X_{t_1}$

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