Fractional Lévy processes

Heikki Tikanmäki
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Introduction

- Fractional Brownian motion (fBM) is a Gaussian process with certain covariance structure
- It has become a popular model in different fields of science, because it allows to model for dependence
- If no Gaussianity assumption, the covariance structure does not define the law uniquely
- There are several ways of defining fractional Lévy processes as generalisations of fBM
- We concentrate on defining fractional Lévy processes (fLP) by integral transformations
- This means that we replace Brownian motion by more general Lévy process in the integral representation of fBM
- FLP's have the same covariance structure as fBM

Fractional Brownian motion

- Fractional Brownian motion (fBM) B^H with Hurst index $H \in (0,1)$ is a zero mean Gaussian process with the following covariance structure

$$\mathbb{E}B_t^H B_s^H = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - 2|t-s|^{2H}\right).$$

- If $H = \frac{1}{2}$, we are in the case of ordinary BM. For $H > \frac{1}{2}$ the process has long range dependence property and for $H < \frac{1}{2}$ the increments are negatively correlated.
- FBM is self-similar with parameter H.
- FBM is not semi-martingale nor Markov process (unless $H = \frac{1}{2}$)

Integral representations of fBM

- A fractional Brownian motion can be represented as an integral of a deterministic kernel w.r.t. an ordinary Brownian motion in two ways.
- Mandelbrot-Van Ness representation of fBM:

$$\left(B_t^H\right)_{t\in\mathbb{R}} \stackrel{d}{=} \left(\int_{-\infty}^{t} f_H(t,s)dW_s\right)_{t\in\mathbb{R}}.$$

- Molchan-Golosov representation of fBM:

$$\left(B_t^H\right)_{t\geq 0} \stackrel{d}{=} \left(\int_{0}^{t} z_H(t,s)dW_s\right)_{t\geq 0}.$$

Integral representation kernels

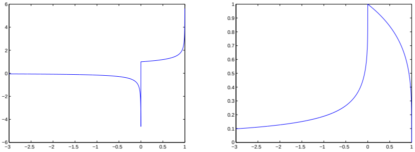

Figure: Mandelbrot-Van Ness kernel with $H = 0.25$ (left) and $H = 0.75$.

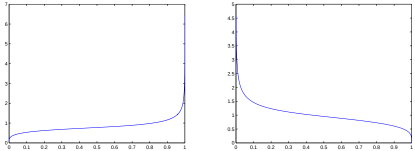

Figure: Molchan-Golosov kernel with $H = 0.25$ (left) and $H = 0.75$.

FLP's by integral transformations

- The main idea is to integrate one of the fBM integral representation kernels w.r.t. a more general square integrable Lévy process.
- We call these processes fractional Lévy procesesses.
- These processes have the same covariance structure as fBM.
- However, different kernels lead to different processes
  - Fractional Lévy processes by Mandelbrot-Van Ness representation (fLPMvN)
  - Fractional Lévy processes by Molchan-Golosov representation (fLPMG)

FLPMvN

Fractional Lévy processes by (infinitely supported) Mandelbrot-Van Ness kernel representation are defined as

$$(X_t)_{t\in\mathbb{R}} \stackrel{d}{=} \left(\int_{-\infty}^{t} f_H(t,s)dL_s \right)_{t\in\mathbb{R}} .$$

- L is a zero mean square integrable Lévy process without Gaussian component.
- Integral can be understood as a limit in probability of elementary integrals, in L^2 sense or pathwise.

Fractional Lévy processes by Mandelbrot-Van Ness representation have been studied by Benassi & al (2004) and Marquardt (2006).

FLPMG

Fractional Lévy processes by (compactly supported) Molchan-Golosov representation are defined as

$$(Y_t)_{t\geq 0} \stackrel{d}{=} \left(\int_0^t z_H(t,s) dL_s \right)_{t\geq 0}.$$

- L is zero mean square integrable Lévy process without Gaussian component as before
- Integral can be understood as a limit in probability of elementary integrals, in L^2 sense and in some cases also pathwise.

The definition in this generality is new to the best of my knowledge.

Paths of different fLP's

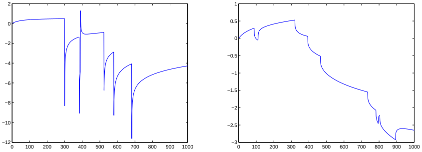

Figure: Sample path of fLPMvN with $H = 0.25$ (left) and $H = 0.75$.

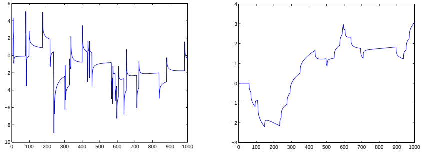

Figure: Sample path of fLPMG with $H = 0.25$ (left) and $H = 0.75$.

Properties of fLPMG

- Hölder continuous paths of any order $\gamma < H - \frac{1}{2}$
- Zero quadratic variation for $H > \frac{1}{2}$
- Discontinuous and unbounded paths with positive probability when $H < \frac{1}{2}$
- Inifinitely divisible law
- Adapted to the natural filtration of driving Lévy process
- Nonstationary increments in general
- Covariance structure of fBM
- Stochastic integration
  - Wiener integrals for deterministic integrands
  - Skorokhod type integration

Comparison of various definitions

Property / Process	fLPMvN	fLPMG
Covariance structure of fBM	Yes	Yes
Stationarity of increments	Yes	No
Adapted (natural filtration)	No	Yes
Pathwise construction for $H > \frac{1}{2}$	Yes	Partial result
Hölder ontinuous paths for $H > \frac{1}{2}$	Yes	Yes
Self-similarity	No	No
Definition does NOT need two-sided processes	No	Yes

Table: Comparison of various definitions of fractional Lévy processes.

Connection of different fLP concepts

$$Y_t^s = \int_0^t z_H(t,u)dL_{u-s}, \quad t \in [0,\infty),$$

is fLPMG with Hurst parameter H. Define the time shifted process

$$Z_t^s = Y_{t+s}^s - Y_s^s, \quad t \in [-s,\infty).$$

Let now

$$Z_t^\infty = c_H X_t = c_H \int_{-\infty}^t f_H(t,v)dL_v, \quad t \in \mathbb{R}$$

be appropriately renormalised fLPMvN. Then we have the following result (analogous to fBM case in Jost (2008))

Theorem

For every $t \in \mathbb{R}$ there exist constants $S, C > 0$ such that

$$\mathbb{E}\left(Z_t^s - Z_t^\infty\right)^2 \leq Cs^{2H-2}, \quad \textit{for } s > S.$$

Financial application

- Fractional Lévy processes (by any of the two transformations) have zero quadratic variation property when $H > \frac{1}{2}$.
- Thus we can use the results of Bender & al (2008) and obtain a no-arbitrage theorem for mixed model where the price of an asset is given by

$$S_t = \exp\left(\epsilon W_t + \sigma Z_t\right),$$

  where W is an ordinary Brownian motion and Z is either fLPMvN or fLPMG with $H > \frac{1}{2}$.
- The model can be used for capturing random shocks in the market that have some long term impacts

References

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Thanks for your attention

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