Diffusion approximation of Lévy processes

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Joint work with R. Tempone.

- We want to calculate $E[g(X_T)]$ using Monte Carlo, when X_t is some infinite activity Lévy process.
- Problem: We can only simulate an approximate finite activity process $\overline{X}_t$.

Questions:

- How do we choose $\overline{X}_t$?
- What is the model error:

$$\mathcal{E} = E[g(X_T)] - E[g(\overline{X}_T)]\,?$$

Outline

1. Motivation
2. Classical results
3. Problems with classical results
4. New results – problems resolved
5. Adaptive schemes

Motivation

Infinite activity Lévy processes are becoming increasingly popular in option pricing.
They have many desirable properties, such as heavy tails, discontinuous trajectories and good ability to reproduce observed option prices.

Setup

- In this talk we assume, for notational ease, that all processes are 1 dimensional. All results extend to higher dimensions.

Recall

- Associated to a Lévy process X_t is a jump measure ν.
- The quantity

$$\nu(A), \quad A \subset \mathbb{R}$$

is the expected number of jumps of size A.
- If $\nu(\mathbb{R}) < \infty$ then X_t is said to have finite activity.

Recall

A finite activity Lévy process X_t is a compound poisson process with added diffusion:

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} J_i$$

where

- γ is the drift, W_t standard Brownian motion.
- The J_i are i.i.d. with law $\nu(\mathrm{d}x)/\nu(\mathbb{R})$.
- N_t is Poisson with parameter $t\nu(\mathbb{R})$.
- $\nu(\mathbb{R})$ the jump intensity.

Definition

The *work* of simulating X_t is the expected number of jumps:

$$\mathrm{Work}\ (X_t) = E[N_t] = t\,\nu(\mathbb{R})$$

If X_t is an infinite activity Lévy process, $\nu(\mathbb{R}) = \infty$, then for every $\epsilon > 0$

$$X_t = X_t^\epsilon + R_t^\epsilon$$

where

- X_t^ϵ has finite activity with jump measure $\nu^\epsilon = 1_{|x|>\epsilon}\nu$:

$$X_t^\epsilon = \gamma^\epsilon t + \sigma W_t + \sum^{N_t} J_i 1_{|J|>\epsilon}.$$

- R_t^ϵ is a pure jump process with jump measure $1_{|x|<\epsilon}\nu$ and

$$E[R_t^\epsilon] = 0.$$

First approximation

Fix an infinite activity Lévy process X_t and some $\epsilon > 0$.

$$X_t \approx X_t^\epsilon \quad \text{i.e.} \quad R_t^\epsilon \approx 0$$

Note that

$$\text{Work } (X_t^\epsilon) = t\,\nu(|x| > \epsilon)$$

Theorem (Jensen's inequality)

If $|g'(x)| \leq C$ *then*

$$\mathcal{E} = \Big| E[g(X_T)] - E[g(X_T^\epsilon)] \Big| \leq C\sigma(\epsilon)\sqrt{T}$$

where

$$\begin{aligned}\sigma^2(\epsilon) &= \int_{|x|<\epsilon} x^2 \nu(dx) \\ &= Var\ R_T^\epsilon / T\end{aligned}$$

Second approximation

[Assmussen & Rosinski '01] If there are "enough" jumps, then

$$R_t^\epsilon/\sigma(\epsilon) \to W_t$$

as $\epsilon \to 0$, in distribution.

Definition

For some fixed $\epsilon > 0$ we define

$$\overline{X}_t = X_t^\epsilon + \sigma(\epsilon) W_t$$

that is, we approximate: $R_t^\epsilon \approx \sigma(\epsilon) W_t$.

Theorem (Berry-Essen type result)

If $|g'(x)| \le C$ *then*

$$\mathcal{E} = \left| E[g(X_T)] - E[g(\overline{X}_T)] \right| \le 16.5C \int_{|x|<\epsilon} |x|^3 \nu(dx)/\sigma^2(\epsilon).$$

Problems with classical results

- Many contracts have payoff with unbounded derivative, e.g. digital options

$$g(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

- These error estimates are independent of the initial value of X_t. It is reasonable to assume that an option far into the money is less sensitive to approximations then an option at the money.

Results

Let X_t be a Lévy process such that there is a $\beta \in (0, 2)$ such that

$$\int_{|x|<\epsilon} x^2 \nu(\mathrm{d}x) = \mathcal{O}(\epsilon^{\beta}) \quad \text{as} \quad \epsilon \to 0,$$

then

Theorem (K. & Tempone)

The model error can be expressed as

$$\begin{aligned} \mathcal{E} &= E[g(X_T)] - E[g(\overline{X}_T)] \\ &= \frac{T}{6} \int_{|x|<\epsilon} x^3 \nu(dx) E[g^{(3)}(\overline{X}_T)] + \mathcal{O}(\epsilon^{2+\epsilon}). \end{aligned}$$

Example

Suppose that X_t is a pure jump process with $E[X_t] = 0$ and jump measure given by

$$\nu(\mathrm{d}x) = \frac{1}{x^2} 1_{0<x<1}.$$

Suppose further that the payoff $g(x)$ is given by

$$g(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

From the above Theorem:

$$\mathcal{E} \approx \frac{T}{12} \epsilon^2 E[\delta''(\overline{X}_T)].$$

- To first order, $E[\delta''(\overline{X}_T)]$ is independent of the choice of ϵ.
- To estimate $E[\delta''(\overline{X}_T)]$ we let $\epsilon = 1$, i.e. all jumps have been replaced by diffusion.
- $\delta''(x)$ is approximated with a difference quotient.
- Note that the work of simulating $\overline{X}_T$ is equal to

$$\mathrm{Work}\ (\overline{X}_T) = T\left(\frac{1}{\epsilon} - 1\right)$$

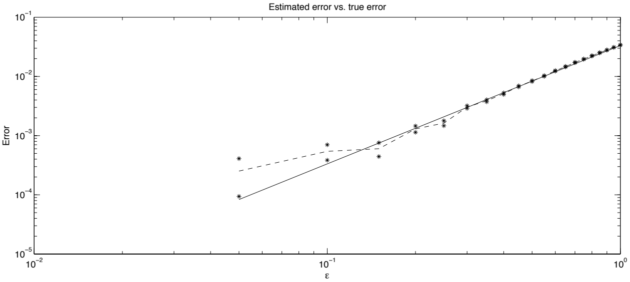

Figure: Here the leading order error term is compared with the true error, estimated with Monte Carlo and a small value of ϵ. In this picture the true error is displayed with a dashed line. The solid line represents the error estimated from the leading term. The dotted lines represent bounds of the statistical error corresponding to one standard deviation.

More results

We can also derive error estimates for

- Barrier options.
- Adaptive schemes.

The goal of an adaptive scheme is to achieve same level of accuracy with less work.

A simple adaptive scheme

- Recall that the model error is proportional to $E[g^{(3)}(\overline{X}_T)]$.
- Fix a *critical region* $L \subset \mathbb{R}$.
- Fix $\epsilon_1 > \epsilon_2 > 0$.
- Define the *adaptive approximation* $\overline{X}_T^{(a)}$ of X_T by:

$$\overline{X}_T^{(a)} = \begin{cases} X_T^{\epsilon_1} + \sigma(\epsilon_1) W_T & \text{if } X_T^{\epsilon_1} \notin L \\ \\ X_T^{\epsilon_2} + \sigma(\epsilon_2) W_T & \text{if } X_T^{\epsilon_1} \in L \end{cases}$$

Error estimates & work

Theorem (K. & Tempone)

The model error is

$$\begin{aligned}\mathcal{E} &= E[g(X_T)] - E[g(\overline{X}_T^{(a)})] \\ &= \frac{T}{6}\Big(\int_{|x|<\epsilon_1} x^3\nu(dx)E\Big[1_{X_T^{\epsilon_1}\notin L}g^{(3)}(\overline{X}_T)\Big] \\ &+ \int_{|x|<\epsilon_2} x^3\nu(dx)E\Big[1_{X_T^{\epsilon_1}\in L}g^{(3)}(\overline{X}_T)\Big]\Big)\end{aligned}$$

The work of simulating the adaptive approximation becomes:

$$\text{Work } (\overline{X}_T^{(a)}) = T\Big(\nu(|x| > \epsilon_1) + \mathbb{P}(X_T^{\epsilon_1} \in L)\nu(\epsilon_2 < |x| < \epsilon_1)\Big)$$

Adaptive vs. standard approximation, an example

- Assume same setup as before, i.e. pure jump process X_t with jump measure $1/x^2 1_{x>0}$. We let the contract be a digital option.
- We compare a particular choice of the adaptive approximation with the non-adaptive approximation by, for each tolerance TOL, comparing the work.

Work comparison adaptive vs. non–adaptive