Spline approximation of random processes with singularity

Konrad Abramowicz

Department of Mathematics and Mathematical Statistics
Umeå university

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Coauthor:

Oleg Seleznjev
Department of Mathematics and Mathematical Statistics

Umeå university

Outline

Suppose a random process $X(t), t \in [0, 1]$, with finite second moment is observed in a finite number of points (**sampling designs**). At any unsampled point t, we approximate the value of the process. The approximation performance on the entire interval is measured by mean errors. In this talk we deal with two problems:

- Investigating accuracy of such interpolator in mean norms
- Constructing a sequence of sampling designs with asymptotically optimal properties

Basic notation

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We set $||\xi|| = \left(\mathrm{E}\xi^2\right)^{1/2}$ for all $\xi \in L^2(\Omega)$ and consider the approximation based on the normed linear space $\mathcal{C}^m[0,1]$ of random processes having continuous q.m. (quadratic mean) derivatives up to order $m \geq 0$.

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We define the **integrated mean norm** for any $X \in \mathcal{C}^m[0, 1]$ by setting

$$||X||_p = \left(\int_0^1 ||X(t)||^p dt\right)^{1/p}, \qquad 1 \leq p < \infty,$$

and the **uniform mean norm** $||X||_\infty = \max_{[0,1]} ||X(t)||$.

Hölder's conditions and local stationarity

We define the classes of processes used throughout the paper. Let $X \in \mathcal{C}^m[a,b]$. We say that

i) $X \in \mathcal{C}^{m,\beta}([a,b],C)$ if $Y(t) = X^{(m)}(t)$ is *Hölder continuous*, i.e., if there exist $0 < \beta \leq 1$ and a positive constant C such that, for all $t, t+s \in [a,b]$,

$$|| \, Y(t+s) - Y(t) \, || \leq C|s|^{\beta}, \tag{1}$$

ii) $X \in \mathcal{V}^{m,\beta}([a,b], V(\cdot))$ if $Y(t) = X^{(m)}(t)$ is *locally Hölder*, i.e., if there exist $0 < \beta \leq 1$ and a positive continuous function $V(\cdot)$ such that, for all $t, t+s, \in [a,b]$, $s > 0$,

$$|| \, Y(t+s) - Y(t) \, || \leq V(t)^{1/2} |s|^{\beta}, \tag{2}$$

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iii) $X \in \mathcal{B}^{m,\beta}([a,b],c(\cdot))$ if $Y(t) = X^{(m)}(t)$ is *locally stationary* (see, Berman (1974)), i.e., if there exist $0 < \beta \leq 1$ and a positive continuous function $c(t)$ such that

$$\lim_{s \to 0} \frac{|| \, Y(t+s) - Y(t) \, ||}{|s|^{\beta}} = c(t)^{1/2} \text{ uniformly in } t \in [a,b]. \tag{3}$$

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We say that $X \in \mathcal{BV}^{m,\beta}((0,1], c(\cdot), V(\cdot))$ if $X \in \mathcal{C}^m[a,b]$ and its m-th q.m. derivative satisfies (2) and (3) for any $[a,b] \subset (0,1]$.

Composite Splines

For any $f \in C^l[0,1]$, $l \geq 0$, the piecewise Hermite polynomial $H_k(t) := H_k(f, T_n)(t)$, of degree $k = 2l+1$, $l \geq 0$, is the unique solution of the interpolation problem $H_k^{(j)}(t_i) = f^{(j)}(t_i)$, where $i = 0, \dots, n$, $j = 0, \dots, l$. Define $H_{q,k}(X, T_n)$, $q \leq k$, to be a *composite Hermite spline*

$$H_{q,k}(X, T_n) := \begin{cases} H_q(X, T_n)(t), & t \in [0, t_1] \\ H_k(X, T_n)(t), & t \in [t_1, 1] \end{cases} .$$

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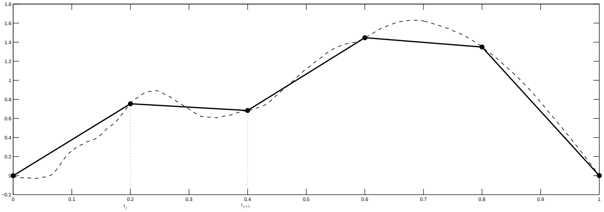

quasi Regular Sequences

We consider **quasi regular sequences** (qRS) of sampling designs $\{T_n = T_n(h)\}$ generated by a density function $h(\cdot)$ via

$$\int_0^{t_i} h(t)dt = \frac{i}{n}, \quad i = 1, \ldots, n,$$

where $h(\cdot)$ is continuous for $t \in (0, 1]$ and if $h(\cdot)$ is unbounded in $t = 0$, then $h(t) \to +\infty$ as $t \to +0$. We denote this property of $\{T_n\}$ by: $\{T_n\}$ is qRS(h). Observe that if $h(\cdot)$ is positive and continuous on $[0, 1]$, then we obtain **regular sequences**.

Denote the distribution function by $H(t) = \int_0^t h(v)dv$, and the quantile function by $G(s) = H^{-1}(s)$, and $g(s) = G'(s)$, $t, s \in [0, 1]$, i.e., $t_j = G(j/n), j = 0, \ldots, n$.

Previous Results

- (Seleznjev, Buslaev 1999)
  Optimal approximation rate for linear methods for $X \in \mathscr{C}^{0,\beta}[0,1]$ is $n^{-\beta}$
- (Seleznjev, 2000)
  Results on Hermite spline approximation when $X \in \mathscr{B}^{m,\beta}([0,1], c(\cdot))$ and regular sequences of sampling designs are used

$$||X - H_k(X, T_n)|| \sim n^{-(m+\beta)} \text{ as } n \to \infty, \quad m \le k.$$

Processes of interest

Let $X(t)$, $t \in [0,1]$, be a stochastic process which $l-th$ q.m. derivatice satisfies Hölder's condition on $[0,1]$ with $0 < \alpha \leq 1$. Moreover the process is q.m. differentiable up to order m on the left-open interval $(0,1]$. The $m-th$ derivative is locally Hölder with $0 < \beta \leq 1$ on $[0,1]$ and locally stationary on any $[a,b] \subset (0,1]$ with β. We denote it by

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- $X_1(t) = Y(t^{1/2})$, $t \in [0,1]$, where $Y(t)$, $t \in [0,1]$, is a fractional Brownian motion with Hurst parameter H,
- $X_2(t) = t^{9/10} Y(t)$, $t \in [0,1]$, where $Y(t)$, $t \in [0,1]$, is a zero mean stationary process with $Cov(Y(t), Y(s)) = exp(-(t-s)^2)$.

Problem formulation

We have a process which $l-th$ derivative is α-Hölder on $[0,1]$.
Can we get the approximation rate **better** than $n^{-(l+\alpha)}$?

Regularly varying function

A positive function $f(\cdot)$ is called **regularly varying** (on the right) at the origin with index ρ, if for any $\lambda > 0$,

$$\frac{f(\lambda x)}{f(x)} \to \lambda^{\rho} \quad \text{as } x \to 0+,$$

and denote this property by $f \in \mathscr{R}_{\rho}(O+)$.

Assumptions and conditions

Let us recall the notation: $H(t) = \int_0^t h(v)dv$, $G(s) = H^{-1}(s)$, and $g(s) = G'(s)$, $t, s \in [0, 1]$.

Assumptions and conditions

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We formulate the following assumptions about the local Hölder function and the sequence generating density:

(A1) there exists a function $r \in \mathcal{R}_\rho(0+), \rho > 0$, such that $r(s) \geq g(s), s \in [0, a]$, for some $a > 0$,

(A2) there exists a function $R \in \mathcal{R}_\rho(0+), \rho > 0$, such that $R(H(t)) \geq V(t)^{1/2}, t \in [0, a]$, for some $a > 0$.

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Moreover we formulate the following conditions on the behavior of the introduced functions in a neighborhood of zero

(C1) the assumption (A1) holds with $r(\cdot)$ such that $V(t)r(H(t))^{2(m+\beta)} \to 0$ as $t \to 0$,

(C2) for a given $1 \leq p < \infty$, the assumptions (A1) and (A2) hold with $r(\cdot)$ and $R(\cdot)$ such that, for some $b > 0$, $R(H(t))r(H(t))^{m+\beta} \in L_p[0, b]$.

Optimal rate recovery

In this section we use the convention, that if $p = \infty$, then $1/p = 0$.

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Theorem

Let $X \in \mathcal{BV}^{m,\beta}((0,1], c(\cdot), V(\cdot)) \cap \mathcal{C}^{l,\alpha}([0,1], M)$, $l + \alpha \leq m + \beta$, *be interpolated by a composite Hermite spline* $H_{q,k}(t, T_n)$, $l \leq q \leq 2l+1$, $m \leq k \leq 2m+1$, *where* T_n *is a qRS(h). Let the condition (C1) be satisfied if* $p = \infty$, *or alternatively, for* $1 \leq p < \infty$, *let the condition (C2) hold. If additionally,*

$$r(s) = o(s^{(m+\beta)/(\alpha+l+1/p)-1}) \text{ as } s \to 0, \tag{4}$$

then

$$\lim_{n\to\infty} n^{m+\beta}||X - H_{q,k}(t, T_n)||_p = b_{k,p}^{m,\beta}||c^{1/2}h^{-(m+\beta)}||_p.$$

For a composite spline $H_{q,k}$, we define **asymptotic optimality** of the sequence of sampling designs T_n^* by

$$\lim_{n\to\infty} ||X - H_{q,k}(X, T_n^*)||_p \Big/ \inf_{T \in D_n} ||X - H_{q,k}(X, T)||_p = 1,$$

where $D_n := \{T_n : 0 = t_0 < t_1 < \ldots < t_n = 1\}$ denotes the set of all $(n+1)$-point designs.

Asymptotically optimal density

Let $\gamma := 1/(m + \beta + 1/p)$, $1 \leq p < \infty$. Define

$$h^*(t) := c(t)^{\gamma/2} / \int_0^1 c(s)^{\gamma/2} ds, \; t \in [0, 1],$$

and denote by $g^*(s)$, $s \in [0, 1]$ the corresponding quantile density function.

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Proposition

Let $X \in \mathcal{BV}^{m,\beta}((0,1], c(\cdot), V(\cdot)) \cap \mathcal{C}^{l,\alpha}([0,1], M)$, $l+\alpha \le m+\beta$, such that $V(t) = O(c(t))$ as $t \to 0$, and $c, V \in \mathcal{R}_\rho(0+)$, be interpolated by a composite Hermite spline $H_{q,k}(t, T_n)$, $l \le q \le 2l+1$, $m \le k \le 2m+1$, where T_n is a qRS(h). If the condition

$$g^*(s) = o(s^{(m+\beta)/(\alpha+l+1/p)-1}) \text{ as } s \to 0, \tag{5}$$

is satisfied, then $h^*(\cdot)$ is asymptotically optimal, and

$$\lim_{n\to\infty} n^{m+\beta} ||X - H_{q,k}(X, T_n(h^*))||_p = b_{k,p}^{m,\beta} ||c^{1/2}||_\gamma .$$

Remark

Theorem and Proposition above extend to the class

$$\mathcal{B}^{m,1}((0,1], c(\cdot)) \cap \mathcal{C}^{l,\alpha}([0,1], M),\ l + \alpha \leq m + 1,$$

with the assumptions and the conditions formulated in terms of local stationarity function $c(\cdot)$ instead of function $V(\cdot)$.

Undersmoothing

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For any $q \leq k < m$ denote $c_k(t) := ||X^{(k+1)}(t)||^2$, and consider the following modification of the assumption (A2),

(A2$'$) there exists a function $R \in \mathcal{R}_\rho(0+), \rho > 0$, such that $R(H(t)) \geq c_k(t)^{1/2}, t \in [0, a]$, for some $a > 0$.

Let us reformulate the conditions in terms of function $c_k(\cdot)$:

(C1$'$) the assumption (A1) holds with $r(\cdot)$, such that $c_k(t)r(H(t))^{2(k+1)} \to 0$ as $t \to 0$,

(C2$'$) for given $1 \leq p < \infty$, the assumptions (A1) and (A2$''$) hold with $r(\cdot)$ and $R(\cdot)$, such that, for some $b > 0$, $R(H(t))r(H(t))^{k+1} \in L_p[0, b]$.

Observe that in this case we do not require the local Hölder property, and the results are formulated for the local stationary class of processes.

Theorem

Let $X \in \mathcal{B}^{m,\beta}((0,1], c(\cdot)) \cap \mathcal{C}^{l,\alpha}([0,1], M)$, $l + \alpha \leq m + \beta$, *be interpolated by a composite Hermite spline* $H_{q,k}(t, T_n)$, $l \leq q \leq 2l + 1$, $q \leq k < m$, *where* T_n *is a qRS(h). Let the condition (C1′) be satisfied if* $p = \infty$, *or alternatively, for* $1 \leq p < \infty$, *let the condition (C2′) hold. If additionally,*

$$r(s) = o(s^{(k+1)/(\alpha+l+1/p)-1}) \text{ as } s \to 0,$$

then

$$\lim_{n\to\infty} n^{k+1}||X - H_{q,k}(t, T_n)||_p = b_{k,p}^{k,1}||c_k^{1/2} h^{-(k+1)}||_p.$$

Asymptotically optimal density

We introduce the asymptotically optimal density, when $1 \leq p < \infty$. Let $\gamma_k := 1/(k+1+1/p)$. Define

$$h_k^*(t) := c_k(t)^{\gamma_k/2} / \int_0^1 c_k(s)^{\gamma_k/2} ds,$$

and denote by $g_k^*(s)$, $s \in [0,1]$ the corresponding quantile density function.

Proposition

Let $X \in \mathcal{B}^{m,\beta}((0,1], c(\cdot)) \cap \mathcal{C}^{l,\alpha}([0,1], M)$, $l + \alpha \leq m + \beta$, such that $c_k \in \mathcal{R}_\rho(0+)$, be interpolated by a composite Hermite spline $H_{q,k}(t, T_n)$, $l \leq q \leq 2l+1$, $q \leq k < m$, where T_n is a qRS(h). If the condition

$$g_k^*(s) = o(s^{(k+1)/(\alpha+l+1/p)-1}) \text{ as } s \to 0,$$

is satisfied, then $h^*(\cdot)$ is asymptotically optimal, and

$$\lim_{n\to\infty} n^{k+1} ||X - H_{q,k}(X, T_n(h_k^*))||_p = b_{k,p}^{k,1} ||c_k^{1/2}||_{\gamma_k}.$$

Numerical Experiments and Examples

When choosing the knot distribution, we consider the densities of a form:

$$h_\lambda(t) = \frac{1}{\lambda} t^{\frac{1}{\lambda}-1}.$$

This leads to

$$t_i = \left(\frac{i}{n}\right)^\lambda,$$

and therefore, such densities are called *power densities*.

Example 1

Let $B_H = B_H(t), t \in [0,1]$ denote a fractional Brownian motion with Hurst parameter $0 \leq H \leq 1$, i.e., a zero mean Gaussian process, starting at zero, with covariance function:

$$K_{B_H}(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

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Consider now a time change version of the process B_H,

$$X(t) := B_H(\sqrt{t}),\ t \in [0,1].$$

Then

$$X \in \mathcal{BV}^{0,H}((0,1], c(\cdot), V(\cdot)) \cap \mathcal{C}^{0,H/2}([0,1],1),$$

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where $c(t) = V(t) = (4t)^{-H}$.
Let $H = 0.8$. We measure the error of the **piecewise linear approximation** in mean uniform norm ($p = \infty$). The conditions of Theorem are satisfied if $\lambda > 2H/(H + 2/p)$. In our experiments we consider the following choice of λ:

$$\lambda_{1,\infty} = 1 \quad \lambda_{2,\infty} = 2.1$$

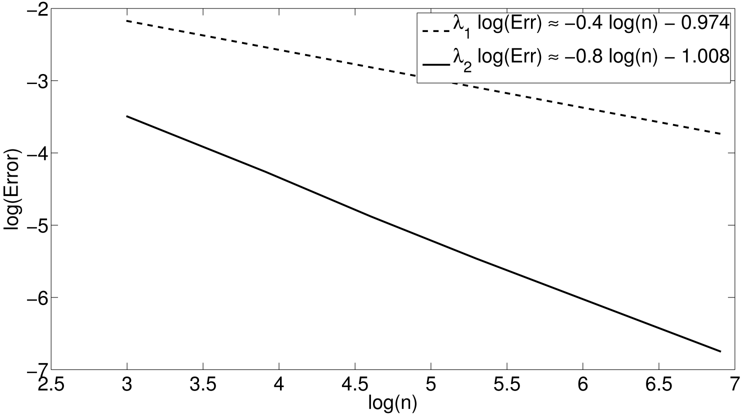

Figure: Comparison of the uniform mean errors for the uniform density $h_{\lambda_{1,\infty}}(\cdot)$ and $h_{\lambda_{2,\infty}(\cdot)}$ in the log-log scale.

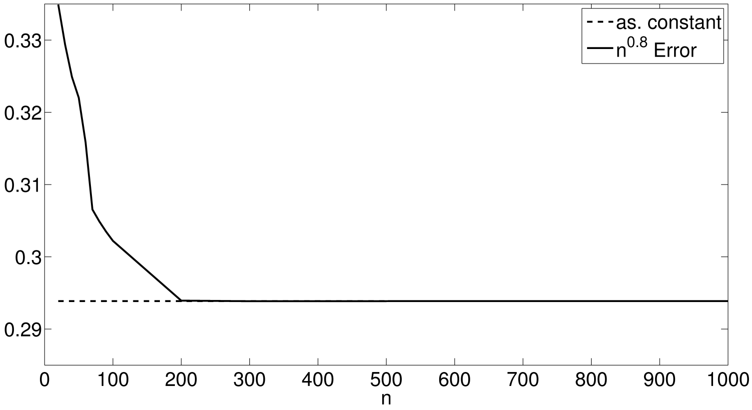

Figure: Convergence of the n^2 scaled uniform mean errors to the asymptotic constant for the generating density $h_{\lambda_{2,\infty}}(\cdot)$.

Example 2

Let $Y(t),\ t \in [0,1]$ be a zero mean Gaussian process with covariance kernel

$$K_Y(s,t) = \exp\{-(s-t)^2\}.$$

Such process has infinitely many q.m. derivatives, hence the rate of convergence of Hermie spline approximation is limited by the order of spline only.

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$$X(t) := t^{0.9} Y(t),\ t \in [0,1].$$

Then

$$X \in \mathcal{B}^{m,1}((0,1], c_m(\cdot)) \cap \mathcal{C}^{0,0.9}([0,1],1), \qquad \text{for any } m \geq 0,$$

where $c_m = ||Y^{m+1}(t)||^2$.

Consider now an approximation by the composite Hermite spline $H_{1,3}$. The conditions of Theorem are satisfied when $\lambda > 4/\left(0.9 + 1/p\right)$, with

$$c_3(t) = 1680t^{\frac{9}{5}} - \frac{44631}{1250}t^{-\frac{11}{5}} - \frac{16929}{125000}t^{-\frac{21}{5}} + \frac{8424}{5}t^{-\frac{1}{5}} + \frac{4322241}{10^8}t^{-\frac{31}{5}}$$

To evaluate mean integrated error ($p = 2$) we consider the following choice of the generating density parameter :

$$\lambda_{1,2} \;= 1 \quad \lambda_{2,2} \;= 3 \quad \lambda_{3,2} = 4 \quad \lambda_{4,2} = 5$$

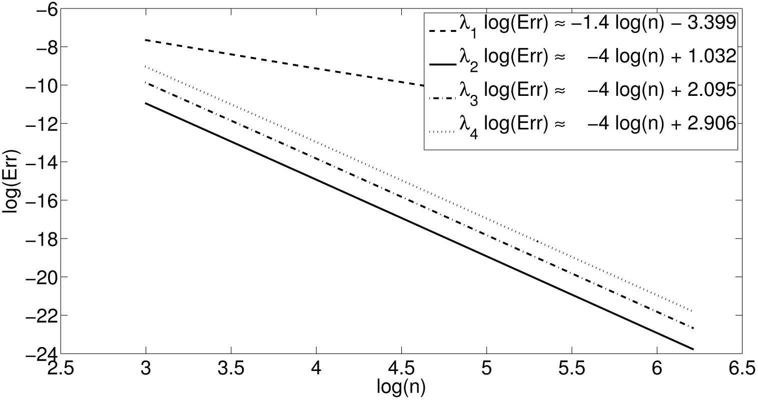

Figure: Comparison of the integrated mean errors, ($p = 2$), for the uniform density $h_{\lambda_{1,2}}(\cdot)$, $h_{\lambda_{2,2}}(\cdot)$, $h_{\lambda_{2,2}}(\cdot)$, and $h_{\lambda_{4,2}(\cdot)}$ in the log-log scale.

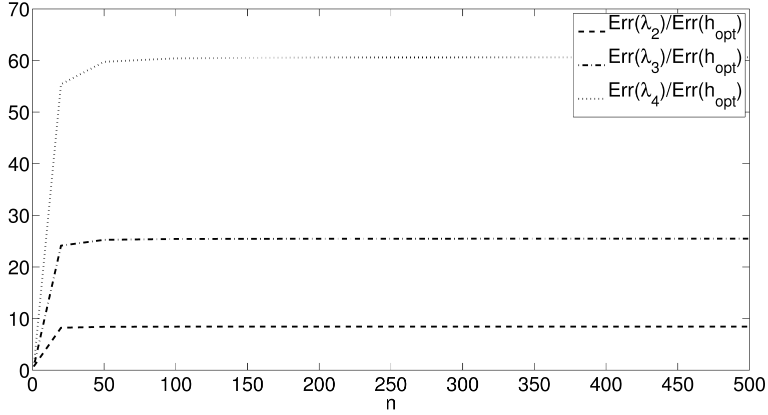

Figure: Ratio of the integrated mean errors, ($p = 2$), for the densities $h_{\lambda_{2,2}}(\cdot)$, $h_{\lambda_{2,2}}(\cdot)$, and $h_{\lambda_{4,2(\cdot)}}$ and the optimal density $h_3^*(\cdot)$.

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