Pricing Bermudan options -
A non parametric estimation approach

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Introduction

• Let \( \{X, Y\} \) be a pair of r.v. such that \( Y \) is square integrable,

• We are seeking an estimator \( f_{D_M}(X) \) of the conditional expectation

\[
 f(X) = \mathbb{E}[Y \mid X]
\]
given a set \( D_M := \{x_m, y_m\}_{m=1}^M \) of independent samples from \( \{X, Y\} \)

Problem: Any parametrized assumption of the solution demand a priori knowledge

• Solution: Smoothed least squares.
Least squares

• Given a priori information about the shape of \( \mathbb{E}[Y|X] \) we may approximate the conditional expectation from the data \( D_M \) by asserting a set of vectors that should hold the key features \( \mathbb{E}[Y|X] \), call them \( \{v_k(X)\}_{k=1}^n \).

• The least squares solution is

\[
\hat{c} = \arg\min_{c \in \mathbb{R}^n} \sum_{m=1}^M \left( y_k - \sum_{k=1}^n c_k v_k(x_k) \right)^2
\]

where

\[
\sum_{k=1}^n \hat{c}_k v_k(X)
\]
Smoothing of least squares solutions

• Assuming no a priori information about the shape of $\mathbb{E}[Y \mid X]$ we are tempted to set $f_{DM}(x_m) = c_m$ where $c \in \mathbb{R}^M$. This gives ill-posedness since then, the least squares solution is $c_m = y_m$, $\forall 1 \leq m \leq M$.

• We may demand for the solution to be sufficiently smooth, by penalizing large deviations between nearby points. The smoothed least squares then looks on the form

$$
\min_{c \in \mathbb{R}^n} \sum_{m=1}^{M} (y_k - f_{DM}(x_k))^2 + \alpha \| \mathcal{A} f_{DM} \|^2
$$

where $\mathcal{A}$ is some operator put on the solution, e.g. the finite difference scheme for a second derivative
Approximation of conditional expectation

- Search for an approximation $f_{DM}(X)$ of the conditional expectation $\mathbb{E}[Y \mid X]$ given data $D_M := \{x_m, y_m\}_{m=1}^M$

- Choose $n$ equidistantly placed points $\{x_k\}_{k=1}^n$ which forms the grid $\Pi$ in the support of $X$ ($\text{supp}\{X\}$)

- Let $A_k := \{x \in \text{supp}\{X\} : \|s_k - x\| \leq \|s_{k'} - x\|, \forall k' \neq k\}$, called the Voronoi teselation of $\Pi$

- Define $f_{DM}(X) := \sum_{k=1}^n c_k I_{A_k}(X)$

- Assumption is the size of the grid
properties of $f_{DM}$

- $f_{DM}$ is a quantization of $\text{supp}\{Y\}$ which has a regularizing property in itself.

- The least squares solution $\hat{f}_{DM}$ gives the coefficients
  \[
  \hat{c}_{k,DM} = \frac{\sum_{m=1}^{M} y_m I_{A_k}(x_k)}{\sum_{m=1}^{M} I_{A_k}(x_k)},
  \]
  needs larges samples to converge.

- Needs to study the linkage between the points $f_{DM}(x_m)$

- Smoothing can be used to connect nearby function values.

- Need to know if convergence still holds.
Convergences of regularized least squares

Proposition 1 Assume that $G(\Pi, \cdot)$ is some operator on the estimator $f_{DM}(X)$ and that $\|G(\Pi, x)\|^2 < \infty$ for any finite $\Pi$ and $x$. Let

$$\sum_{m=1}^{M} \left( y_k - \hat{f}_{DM}(x_k) \right)^2 + \alpha \left\| \mathcal{A} \hat{f}_{DM} \right\|^2 =$$

$$\min \sum_{m=1}^{M} \left( y_k - f_{DM}(x_k) \right)^2 + \alpha \| \mathcal{A} f_{DM} \|^2$$

If $M \to \infty$,

$$\hat{f}_{DM}(x; \Pi, G, \alpha) \to \sum_{k=1}^{n} \mathbb{E} [Y \mid X \in A_k] I_{A_k}(x) \ a.s.,$$

for any $\alpha \in \mathbb{R}^+$. 
Convergents of the conditional expectation

Let \( \Pi \) be a grid in \( A \subseteq \mathbb{R}^d \) and let
\[
\{ A_k \}_{k=1}^n, \quad n \geq 1
\]
be the Voronoi tessellation which is generated by \( \Pi \). Define the outer sets to be the sets \( \{ A_k \}_{k \in N_O} \), where
\[
N_O = \left\{ k \in (1, \ldots, n) : \sup_{x,y \in A_k} \| x - y \| = \infty \right\}, \quad (2)
\]
and define the inner sets to be the sets \( \{ A_k \}_{k \in N_I} \), where
\[
N_I = \left\{ k \in (1, \ldots, n) : \sup_{x,y \in A_k} \| x - y \| < \infty \right\}, \quad (3)
\]
Convergence of the conditional expectation

Proposition 2 In the setting of the previous proposition, let

\[ \{A_{k,n}\}_{k=1}^{n}, \quad n \geq 1 \]

be the Voronoi tessellations on \( \text{supp}(X) \) generated by a sequence of grids \( \Pi_n \). Let \( N_{I,n} \) and \( N_{O,n} \) be the indexes of the inner and outer sets respectively of tessellation \( n \), given by (2) and (3). Assume that

1. \( f \) is twice differentiable, where \( f(X) = \mathbb{E}[Y \mid X] \),
2. \( f \) is locally Lipschitz on \( \{ x \in \mathbb{R}^d : \|x\| > a \} \), for some \( a \),
3. \( P(X \in \bigcup_{k \in N_{O,n}} A_{k,n}) \to 0 \), as \( n \to \infty \),
4. \( \max_{k \in N_{I,n}} \sup_{x,y \in A_{k,n}} \|x - y\| \to 0 \), as \( n \to \infty \).

Then, if we first let \( M \to \infty \), we have that

\[ \hat{f}_D(X; \Pi_n, G, \alpha) \to \mathbb{E}[Y \mid X] \]

in \( L^2 \), as \( n \to \infty \), for any, \( G \) and \( \alpha \).
Specifying $X$ and $Y$

- Consider the Markov process $S$ on the space $\{\Omega, \mathcal{F}, Q\}$ satisfying the SDE

$$dS_t = r_tS_t dt + \sigma_t S_t dW_t^Q$$

where $W_t^Q$ is a BM under the probability measure $Q$, $r_t = 0$ (for convenience) and $\sigma \in \mathbb{R}$

- Our data set $D_M$ consists of $M$ independent samples of $\{S_{T_0}, S_{T_1}, S_{T_2}\}, T_0 < T_1 < T_2$ where $S_{T_0}$ is deterministic.

- Under $Q$ every $T$-claim is an $\mathcal{F}$-martingale, i.e. $V(t, S_T) := \mathbb{E}^Q[\Phi(S_T) \mid \mathcal{F}_t]$ is a $\mathcal{F}_t$-martingale under $Q$
Bermudan contract

- A bermudan contract gives the holder the right to the payoff $\Phi(S_T)$ at one time $T \in \mathcal{T} := \{T_1, T_2\}$ which is up to the holder to decide.

- Optimal stopping problem: the value of the contract is
  
  $$V(t, S_t) = \sup_{\tau \in \mathcal{T}} \mathbb{E}^Q \left[ \Phi(S_\tau) \mid S_t \right], \ t < T_1$$

- Optimal to redeem the contract at times
  
  $$\{\tau \in \mathcal{T} : \Phi(S_\tau) = V(\tau, S_\tau)\}$$
Approach for finding solution

- \( V(T_2, S_{T_2}) = \Phi(S_{T_2}) \)

- \( V(T_1, S_{T_1}) = \max \{ \Phi(S_{T_1}), \mathbb{E}^Q[V(T_2, S_{T_2}) \mid S_{T_1}] \} \)

- \( V(T_0, S_{T_0}) = \mathbb{E}^Q[V(T_1, S_{T_1}) \mid S_{T_0}] \)

- Snell Envelope: Optimal to stop for

\[
\{ \tau \in \mathcal{T} : \Phi(S_{T_{i-1}}) \geq \mathbb{E}[V(T_n, S_{T_n}) \mid S_{T_{n-1}}] \}
\]

- Searching for \( V(T_{i-1}, S_{T_{i-1}}) := \mathbb{E}[V(T_i, S_{T_i}) \mid S_{i-1}] \), where \( V(T_2, S_{T_2}) = \Phi(S_{T_2}) \).
PDE formulation for the solution

Since the value process $V(t, S_t)$ is an $\mathcal{F}_t$-martingale under $Q$, assuming sufficiently nice processes $S_t$, the value process must satisfy

$$(\partial_t + \mathcal{A}) V(t, s) = 0, \quad T_{i-1} < t < T_i$$

where $\mathcal{A}$ is the infinitesimal operator for $S$. 
PDE formulation for the solution

- We want to demand for our estimate \( f_{D_M}(X) \) to satisfy

\[
(\partial_t + \mathcal{A}) f_{D_M^i} = 0,
\]

where \( D_M^i := \{ s_{T_{i-1},m}, s_{T_i,m} \}_{m=1}^M \), to add known information about the solution apart from the simulations.

- We formulate

\[
\hat{c}_{D_M^i} = \arg\min_{c \in \mathbb{R}^n} \sum_{m=1}^M \left( V(T_i, s_{T_i,m}) - f_{D_M^i}(s_{T_{i-1},m}) \right)^2 + \alpha \left\| (\Delta + B) f_{D_M^i} \right\|^2
\]

and \( \hat{f}_{D_M^i}(X) = \sum_{k=1}^n \hat{c}_{k,D_M^i} I_A_k(X) \) where \( \Delta \) is the discretized time derivative, \( B \) is the discretized infinitesimal operator for \( S \). \( \alpha \) is the smoothing parameter.
Bermudan put option example
Comparison: Bermudan put option by a priori vector assumption
Discussion

- Loosen the a priori assumptions
- Introduce relation between PDE and Monte Carlo methods
- Stable in terms of the smoothing parameter $\alpha$
- Not yet investigated how convergence is affected by the smoothing
References

