Option Pricing in Multivariate Stochastic Volatility Models of OU Type

Oliver Pfaffel
(joint work with Johannes Muhle-Karbe and Robert Stelzer)

TUM Institute for Advanced Study
Technische Universität München

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What do we demand on a model for some financial asset?

- Reproduce the **stylized facts**:
  - stochastic volatility
  - volatility exhibits jumps
  - asymmetric and heavy tailed return distribution
  - dependent returns with vanishing autocorrelations

- Analytically tractable

- Calibration to market prices should be possible
Barndorff-Nielsen/Shephard (BNS) model

Model the price process $e^Y$ of some asset by

$$
dY_t = \left( \mu + \beta \sigma^2_t \right) dt + \sigma_t dW_t + \rho dL_t, \quad Y_0 \in \mathbb{R}
$$

$$
d\sigma^2_t = -\lambda \sigma^2_t dt + dL_t, \quad \sigma_0 > 0,
$$

where

- $L$ is a subordinator (i.e. an increasing Lévy process) independent of the Wiener process $W$,
- $\mu, \beta \in \mathbb{R}$,
- mean reversion parameter $\lambda > 0$,
- and leverage parameter $\rho \in \mathbb{R}$.
What if we want to consider options which depend on more than one asset?

We need a multivariate model which satisfies the following:

- Stylized facts / flexibility in the (univariate) marginal models
- Pricing of single-asset options not more involved than in the univariate case
- Reflects the complex interdependencies between several underlying assets (e.g. stochastic correlations)
- Amenable to calibration (high-dimensional optimisation problem)

⇒ We propose such a model
Some notation

We denote by

- $M_d(\mathbb{R})$, $M_d(\mathbb{C})$ the $d \times d$ matrices over $\mathbb{R}$ or $\mathbb{C}$,
- $\mathcal{S}_d$ the symmetric $M_d(\mathbb{R})$-matrices,
- $\mathcal{S}_d^+$ the positive semidefinite matrices in $\mathcal{S}_d$,
- $B^{\frac{1}{2}}$ the positive semidefinite square root of $B \in \mathcal{S}_d^+$, i.e. $B^{\frac{1}{2}} B^{\frac{1}{2}} = B$,
- $\text{tr} (\cdot)$ the trace of a matrix.

- subscripts components of a vector or matrix, but use superscripts for stochastic processes to avoid double indices.
A Multivariate Stochastic Volatility Model of OU Type
Multivariate SV model of OU type

Model the price processes \( e^Y := (e^{Y^1}, \ldots, e^{Y^d}) \) of \( d \) assets jointly by

\[
dY_t = (\mu + \beta(\Sigma_t)) dt + \Sigma_t^{1/2} dW_t + \rho(dL_t), \quad Y_0 \in \mathbb{R}^d
\]
\[
d\Sigma_t = (A\Sigma_t + \Sigma_t A^T) dt + dL_t, \quad \Sigma_0 \in \mathbb{S}_d^+,
\]

where

- \( L \) is a matrix subordinator, i.e. a matrix valued Lévy process such that

\[
L_t - L_s \in \mathbb{S}_d^+ \text{ for all } t \geq s, \quad s, t \in \mathbb{R}^+,
\]

- \( W \) is an \( \mathbb{R}^d \)-valued Wiener process independent of \( L \),

- linear operators \( \beta, \rho : M_d(\mathbb{R}) \to \mathbb{R}^d, \mu \in \mathbb{R}^d \),

- and a mean reversion matrix \( A \in M_d(\mathbb{R}) \) which only has eigenvalues with negative real part.
Marginal dynamics of a single asset

- For example, consider $\beta$, $\rho$ and $A$ diagonal, i.e.

$$
\beta(X) = \begin{pmatrix}
\beta_1 X^{11} \\
\vdots \\
\beta_d X^{dd}
\end{pmatrix}, \quad 
\rho(X) = \begin{pmatrix}
\rho_1 X^{11} \\
\vdots \\
\rho_d X^{dd}
\end{pmatrix}, \quad 
A = \begin{pmatrix}
a_1 & 0 \\
\vdots & \ddots \\
0 & \cdots & a_d
\end{pmatrix}
$$

for $X \in M_d(\mathbb{R})$, with $\beta_1, \ldots, \beta_d, \rho_1, \ldots, \rho_d \in \mathbb{R}$, $a_1, \ldots, a_d < 0$.

- In this case

$$
\begin{align*}
dY_t^1 & \stackrel{\text{fidi}}{=} (\mu_1 + \beta_1 \Sigma_t^{11})dt + \sqrt{\Sigma_t^{11}} dW_t^1 + \rho_1 dL_t^{11} \\
d\Sigma_t^{11} & = 2a_1 \Sigma_t^{11} dt + dL_t^{11}.
\end{align*}
$$

where $\stackrel{\text{fidi}}{=}$ denotes equality of all finite-dimensional distributions.

$\Rightarrow$ BNS model for $Y^1$ (in distribution), analogously for $Y^2, \ldots, Y^d$. 
Martingale Conditions and Equivalent Martingale Measures
Denote the triplet of the matrix subordinator $L$ by $(\gamma_L, 0, \kappa_L)$.

**Theorem (Martingale Conditions)**

The discounted price process $(e^{Y_t - rt})_t \in \mathbb{R}^+$ is a martingale if and only if

$$
\int_{\{X \in S_d^+: \|X\| > 1\}} e^{\rho_i(X)} \kappa_L(dX) < \infty, \quad i = 1, \ldots, d
$$

and

$$
\beta(X) = -\frac{1}{2}(X_{11}, \ldots, X_{dd})^T, \quad X \in M_d(\mathbb{R}),
$$

$$
\mu_i = r - \int_{S_d^+} (e^{\rho_i(X)} - 1) \kappa_L(dX) - \rho_i(\gamma_L), \quad i = 1, \ldots, d,
$$

where $\rho(X)$ denotes the $i$-th component of $\rho(X)$, and $r > 0$ the riskless interest rate.
Equivalent martingale measures

Define the set of EMMs

\[ \mathcal{M} := \{ Q : Q \sim P, (e^{Y_t-r}t)_{t \in \mathbb{R}^+} \text{ is a } Q\text{-martingale} \}. \]

We can characterize all EMMs (if the filtration is generated by \( W \) and \( L \)).

Under some measure \( Q \in \mathcal{M} \)

- \( L \) may not be a Lévy process.
- \( W \) and \( L \) may not be independent.

\text{→ We do not stay in the same model!}
\text{→ But there exists a subclass that preserves the structure of our model.}
Structure preserving EMMs

Consider a finite time interval $[0, T]$. Suppose for simplicity $\gamma_L = 0$. Let $y : S^+_d \to \mathbb{R}^{++}$ be such that

1. $\int_{S^+_d} (\sqrt{y(X)} - 1)^2 \kappa_L(dX) < \infty$,
2. $\int_{\{X \in S^+_d : ||X|| > 1\}} e^{\rho^i(X)} \kappa^y_L(dX) < \infty$ for all $i = 1, \ldots, d$,

where $\kappa^y_L(B) := \int_B y(X) \kappa_L(dX)$ for $B \in \mathcal{B}(S^+_d)$. Define

$$
\psi_t := -\Sigma_t^{-\frac{1}{2}} \left( \mu + \beta(\Sigma_t) + \frac{1}{2} \left( \begin{array}{ccc} \Sigma_{11}^t \\ \vdots \\ \Sigma_{dd}^t \end{array} \right) \right) + \left( \begin{array}{c} \int_{S^+_d} (e^{\rho^1(X)} - 1) \kappa^y_L(dX) \\ \vdots \\ \int_{S^+_d} (e^{\rho^d(X)} - 1) \kappa^y_L(dX) \end{array} \right) - 1r,
$$

where $1 = (1, \ldots, 1)^T \in \mathbb{R}^d$. 
Then

- $Z = \mathcal{E} \left( \int_0^\cdot \psi_s \, dW_s + (y - 1) \ast (\mu^L - \nu^L) \right)$ is a density process,
- the probability measure $Q$ defined by $\frac{dQ}{dP} = Z_T$ is an EMM,
- $W^Q := W - \int_0^\cdot \psi_s \, ds$ is a $Q$-standard Brownian motion,
- $L$ is a $Q$-matrix subordinator with Lévy-Khintchine triplet $(0, 0, \kappa^Y_L)$,
- $L$ and $W^Q$ are independent,
- and the dynamics under $Q$ are given by

\[
\begin{align*}
    dY_t^i &= \left( r - \int_{S^+_d} (e^{\rho^i(X)} - 1) \kappa^Y_L(dX) - \frac{1}{2} \Sigma^i_t \right) dt + \left( \Sigma^\frac{1}{2}_t \, dW^Q_t \right)^i + \rho^i(dL_t), \\
    d\Sigma_t &= (A\Sigma_t + \Sigma_t A^T) \, dt + dL_t.
\end{align*}
\]

for all $i = 1, \ldots, d$. 

Oliver Pfaffel (TU München)  
Pricing in MSV models of OU type  
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13 / 34
Characteristic Function and Option Pricing
A heuristic motivation of Fourier Pricing

- Payoff: $f(Y_T)$, for example a basket option with
  $f(Y_T) = (w_1 e^{Y^1_T} + \ldots + w_d e^{Y^d_T} - K)^+$

- Fourier inversion (for suitable $R \in \mathbb{R}^d$):

$$f(Y_T) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(iR-u)^T Y_T} \hat{f}(iR - u) \, du$$

- Interchanging expectation with the integral above yields the Fourier Pricing Formula:

$$E_Q(e^{-rT} f(Y_T)) = \frac{e^{-rT}}{(2\pi)^d} \int_{\mathbb{R}^d} \Phi_{Y_T}(R + iu) \hat{f}(iR - u) \, du$$  \hspace{1cm} (1)

where $\Phi_{Y_T}(R + iu) = E_Q(e^{(R+iu)^T Y_T})$.

⇒ Formula for $\Phi_{Y_T}$? When does (1) hold rigorously? Conditions on $\Phi_{Y_T}$?
Joint characteristic function of \((Y_t, \Sigma_t)\)

\[
E \left[ e^{i \langle (y, z), (Y_t, \Sigma_t) \rangle} \right] = \exp \left( iy^T (Y_0 + \mu t) + i \text{tr} (\Sigma_0 h_{y,z}(t)) + \int_0^t \Psi_L (h_{y,z}(s) + \rho^*(y)) \, ds \right),
\]

where

\[
h_{y,z}(s) = e^{A^T s} \left( z + A^{-*} \left( \beta^*(y) + \frac{i}{2} yy^T \right) \right) e^{A s} - A^{-*} \left( \beta^*(y) + \frac{i}{2} yy^T \right)
\]

and

- \(\langle (y, z), (Y_t, \Sigma_t) \rangle = y^T Y_t + \text{tr} (z^T \Sigma_t)\) for \((y, z) \in \mathbb{R}^d \times M_d(\mathbb{R})\)
  (\(z = 0 \Rightarrow \) characteristic function of \(Y_t\))
- \(\Psi_L\) the characteristic exponent of \(L\), i.e. \(E(e^{i \text{tr}(Z L_1)}) = e^{\Psi_L(Z)}\),
- \(\beta^*, \rho^* : \mathbb{R}^d \to M_d(\mathbb{R})\) are the adjoints of \(\beta, \rho\),
- \(A^{-*}\) is the inverse of \(A^* : M_d(\mathbb{R}) \to M_d(\mathbb{R}), X \mapsto A^T X + XA.\)
Strip of analyticity and absolute integrability of the mgf

Suppose the matrix subordinator $L$ satisfies

$$\int_{\{X \in S^+_d : ||X|| \geq 1\}} e^{\text{tr}(RX)} \kappa_L(dX) < \infty \quad \text{for all } R \in M_d(\mathbb{R}) \text{ with } ||R|| < \epsilon$$

for some $\epsilon > 0$. Then the moment generating function (mgf) $\Phi_{Y_t}(y) := E\left(e^{y^T Y_t}\right)$ of $Y_t$ is (complex) analytic in the open strip

$$S_\theta := \{y \in \mathbb{C}^d : ||\text{Re}(y)|| < \theta\},$$

where

$$\theta := - \frac{||\rho||}{(e^{2||A||_t} + 1)||A^{-*}||} - ||\beta|| + \sqrt{\Delta} > 0,$$

$$\Delta := \left(\frac{||\rho||}{(e^{2||A||_t} + 1)||A^{-*}||} + ||\beta||\right)^2 + \frac{2\epsilon}{(e^{2||A||_t} + 1)||A^{-*}||}.$$
**Fourier pricing**

- Let $Q$ be some EMM
- Choose some $R \in \mathbb{R}^d$ such that $||R|| < \theta$ (i.e. $R + iu \in S_\theta$)
- Multivariate payoff function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $x \mapsto e^{-R^T x} f(x) \in L^1 \cap L^\infty$
- $\hat{f}$: Fourier transform of $f$
- Arbitrage-free $t = 0$ price of an option with payoff $f(Y_T)$ at time $T > 0$:

$$E_Q \left( e^{-rT} f(Y_T) \right) = \frac{e^{-rT}}{(2\pi)^d} \int_{\mathbb{R}^d} \Phi_{Y_T}(R + iu) \hat{f}(iR - u) \, du,$$

(cf. Eberlein, Glau, Papapantoleon [2009], univariate: Carr, Madan [99])
- Examples: Plain call options, power options, basket options, spread options, options on the minimum/maximum,...
Pricing of spread options with zero strike

- Formula above not applicable, because the Fourier transform of a zero-strike spread option does not exist.
- Suppose

\[ \Phi(Y_1^T, Y_2^T)(R, 1 - R) < \infty \quad \text{for some } R > 1. \]

Then the \( t = 0 \) price of a zero-strike spread option with payoff \((e^{Y_1^T} - e^{Y_2^T})^+\) at \( T > 0 \) is given by

\[
E_Q(e^{-rT}(e^{Y_1^T} - e^{Y_2^T})^+) = \frac{e^{-rT}}{2\pi} \int_{\mathbb{R}} \frac{\Phi(Y_1^T, Y_2^T)(R + iu, 1 - R - iu)}{(R + iu)(R + iu - 1)} du
\]
Calibration of the OU-Wishart model in two dimensions
The OU-Wishart model in $d = 2$

\[ d \begin{pmatrix} Y^1_t \\ Y^2_t \end{pmatrix} = \begin{pmatrix} (\mu_1 - \frac{1}{2} \Sigma^1_{t}^{11} ) \\ (\mu_2 - \frac{1}{2} \Sigma^2_{t}^{22} ) \end{pmatrix} dt + \begin{pmatrix} \Sigma^1_{t}^{11} \\ \Sigma^2_{t}^{22} \end{pmatrix}^{\frac{1}{2}} d \begin{pmatrix} W^1_t \\ W^2_t \end{pmatrix} + \begin{pmatrix} \rho_1 dL^1_{t} \\ \rho_2 dL^2_{t} \end{pmatrix} \]

\[ d \begin{pmatrix} \Sigma^1_{t}^{11} \\ \Sigma^1_{t}^{12} \\ \Sigma^2_{t}^{12} \\ \Sigma^2_{t}^{22} \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ 0 \\ 0 \\ \gamma_2 \end{pmatrix} dt + \begin{pmatrix} 2a_1 \Sigma^1_{t}^{11} \\ (a_1 + a_2) \Sigma^1_{t}^{12} \\ (a_1 + a_2) \Sigma^2_{t}^{12} \\ 2a_2 \Sigma^2_{t}^{22} \end{pmatrix} dt + d \begin{pmatrix} L^1_{t}^{11} \\ L^1_{t}^{12} \\ L^2_{t}^{12} \\ L^2_{t}^{22} \end{pmatrix} \]

- Here $L$ is a compound Poisson process with $\mathcal{W}_d(2, \Theta)$ Wishart distributed jumps.
  (i.e. the jumps have the same distribution as $\Theta^{\frac{1}{2}} X X^T \Theta^{\frac{1}{2}}$, where $X$ is a $2 \times 2$ random matrix with independent standard Normal entries).

- Parameters: $\gamma_1, \gamma_2 \geq 0$, $a_1, a_2 < 0$, $\rho_1, \rho_2 \in \mathbb{R}$, $\Theta \in \mathbb{S}_d^+$. 
Option pricing in the OU-Wishart model

- \( W_1(2, \Theta_{11}) = \text{Exponential dist. with mean } 2\Theta_{11} \)
  \( \Rightarrow \Gamma\text{-OU BNS model at the margins (in distribution).} \)
  \( \Rightarrow \text{Efficient single-asset option pricing as in 1-dim model, speeds up calibration significantly} \)

  - In case \( a_1 = a_2 \) (same mean reversion): Closed form expression of the mgf \( \Phi_{YT} \) exists.
  - Otherwise: Numerical evaluation of \( \Phi_{YT} \).

- OU-Wishart model for \( d > 2 \): Calibration possible using only two-asset options, thus only involving at most two-dimensional integrals.
**Modeling the exchange rates of three currencies**

- \( e^{Y_1} = S$/€: Price of 1 \( € \) in $ (EUR/USD exchange rate).
- \( e^{Y_2} = S$/£: Price of 1 \( £ \) in $ (GBP/USD exchange rate).
- \( r_s, r_€, r_£ \) riskless interest rate in each currency. Martingale conditions:

\[
\mu_1 = r_s - r_€ - \frac{\lambda \rho_1}{2 \Theta_{11} - \rho_1}, \quad \mu_2 = r_s - r_£ - \frac{\lambda \rho_2}{2 \Theta_{22} - \rho_2}
\]

- Getting a multi-asset option on \( S$/€ \) and \( S$/£:
  - \( S/^€\): Price of 1 \( € \) in \( £ \) (EUR/GBP exchange rate).
  - Call Option on \( S/^€ \) has \( £ \)-payoff: \( (S_T/^€ - K)^+ \), i.e. $-payoff

\[
S_T^$/£ \left( S_T/^€ - K \right)^+ = \left( S_T^$/€ - KS_T^$/£ \right)^+
\]

\( \Rightarrow \) Zero-strike spread option on \( S$/€ \) and \( S$/£.\)
Calibration data and results

- Mid-prices of 278 European call options on EUR/USD, GBP/USD and EUR/GBP for different strikes and maturities
- Issued by BNP Paribas and DZ Bank, listed on EUWAX.
- Riskless interest rates: 3 month LIBOR for each currency.
- Minimise the mean squared error (MSE) of market and model prices.
- Runtime: 6 hours.
  \[ \text{RMSE: } 0.0586; 0.0683 \text{ and } 0.0425 \text{ at the marginals.} \]
- For comparison: 0.0610 and 0.0320 in unrestricted univariate calibration.
- Assuming same mean reversion \( a_1 = a_2 \):
  - Runtime: 2 hours.
  - RMSE: 0.0591; 0.0686 \text{ and } 0.0439 \text{ at the margins.}
Fit to EUR/USD Call prices

Call on EUR/USD with maturity 58 days

Call on EUR/USD with maturity 449 days

market prices = ’O’, model prices = ’+’
Fit to GBP/USD Call prices

market prices = 'O',  model prices = '+'
Fit to EUR/GBP Call prices

\[ \Rightarrow \] Good fit across different strikes and maturities
**Implied Volatility Plots**

Market prices = dot, model prices = solid line

EUR/GBP, 129 days

EUR/GBP, 200 days

EUR/GBP, 270 days

EUR/GBP, 391 days
Pricing of Covariance Swaps

- Nontrivial correlation structure ⇒ consider covariance swaps, i.e. contracts with payoff $[Y^1, Y^2]_T - K$ with covariance swap rate $K = E([Y^1, Y^2]_T)$.

- Closed form expression for the covariance swap rate in the OU-Wishart model:

$$K = \frac{1}{a_1 + a_2} \left[ \left( e^{(a_1+a_2)T} - 1 \right) \left( \Sigma_{12}^0 + \frac{\lambda n \Theta_{12}}{a_1 + a_2} \right) - T \lambda n \Theta_{12} \right] + T \rho_1 \rho_2 \lambda n \left( 2 \Theta_{12}^2 + n \Theta_{11} \Theta_{22} \right)$$
Thank you!
References

Muhle-Karbe, J., Pfaffel, O. and Stelzer, R.
Option Pricing in Multivariate Stochastic Volatility Models of OU Type.
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Eberlein, E., Glau, K. and Papapantoleon, A.
## Calibration: Resulting parameters

**Different mean reversion**

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**Same mean reversion**

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Closed form expression for $\Phi_{Y_t}$ in the OU-Wishart model

\[
E[e^{y^T Y_t}] = \exp \left\{ y^T (Y_0 + \mu t) + \frac{e^{2at} - 1}{2a} \text{tr} \left( \Sigma_0 \left( \beta^*(y) + \frac{1}{2}yy^T \right) \right) \\
+ \frac{1}{4a^2} (e^{2at} - 1) \text{tr} \left( \gamma_L \left( \beta^*(y) + \frac{1}{2}yy^T \right) \right) \\
+ t \text{tr} \left( \gamma_L \left( \rho^*(y) - \frac{1}{2a} \left( \beta^*(y) + \frac{1}{2}yy^T \right) \right) \right) \\
+ \frac{\lambda}{2ab_0} \left[ \frac{b_1}{d} \left( \arctan \left( \frac{2b_2 + b_1}{d} \right) \right) - \arctan \left( \frac{2b_2 e^{2at} + b_1}{d} \right) \right] \\
+ \frac{1}{2} \ln \left( \frac{b_0 + b_1 + b_2}{b_2 e^{4at} + b_1 e^{2at} + b_0} \right) \right\} + \frac{\lambda}{b_0} t - \lambda t ,
\]

($b_0, b_1, b_2, d$ depend on $y$)
Closed form expression for $\Phi_{Y_t}$ in the OU-Wishart model

where

\[
\begin{align*}
a & := a_1 = a_2 \\
b_0 & := 1 + 4 \det(B - C) + 2 \text{tr}(B - C) \\
b_1 & := -8 \det(B) + 4 \text{tr}(B) \text{tr}(C) - 4 \text{tr}(BC) - 2 \text{tr}(B) \\
b_2 & := 4 \det(B) \\
d & := \sqrt{4b_0b_2 - b_1^2},
\end{align*}
\]

and

\[
\begin{align*}
B & := \frac{1}{2a} \left( \beta^*(y) + \frac{1}{2} yy^T \right) \Theta \\
C & := \rho^*(y) \Theta.
\end{align*}
\]

⇒ Formula is easy to implement.
⇒ Option prices can be computed numerically efficient.