Option Pricing in Multivariate Stochastic Volatility Models of OU Type

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What do we demand on a model for some financial asset?

- Reproduce the stylized facts:
  - stochastic volatility
  - volatility exhibits jumps
  - asymmetric and heavy tailed return distribution
  - dependent returns with vanishing autocorrelations
- Analytically tractable
- Calibration to market prices should be possible

Barndorff-Nielsen/Shephard (BNS) model

Model the price process e^Y of some asset by

$$\begin{aligned} dY_t &= (\mu + \beta\sigma_t^2)dt + \sigma_t dW_t + \rho dL_t, \quad Y_0 \in \mathbb{R} \\ d\sigma_t^2 &= -\lambda\sigma_t^2 dt + dL_t, \quad \sigma_0 > 0, \end{aligned}$$

where

- L is a subordinator (i.e. an increasing Lévy process) independent of the Wiener process W,
- $\mu, \beta \in \mathbb{R}$,
- mean reversion parameter $\lambda > 0$,
- and leverage parameter $\rho \in \mathbb{R}$.

What if we want to consider options which depend on more than one asset?

We need a multivariate model which satisfies the following:

- Stylized facts / flexibility in the (univariate) marginal models
- Pricing of single-asset options not more involved than in the univariate case
- Reflects the complex interdependencies between several underlying assets (e.g. stochastic correlations)
- Amenable to calibration (high-dimensional optimisation problem)

$\Rightarrow$ We propose such a model

Some notation

We denote by

- $M_d(\mathbb{R})$, $M_d(\mathbb{C})$ the $d \times d$ matrices over $\mathbb{R}$ or $\mathbb{C}$,
- $\mathbb{S}_d$ the symmetric $M_d(\mathbb{R})$-matrices,
- $\mathbb{S}_d^+$ the positive semidefinite matrices in $\mathbb{S}_d$,
- $B^{\frac{1}{2}}$ the positive semidefinite square root of $B \in \mathbb{S}_d^+$, i.e. $B^{\frac{1}{2}} B^{\frac{1}{2}} = B$,
- $\mathrm{tr}(\cdot)$ the trace of a matrix.
- subscripts components of a vector or matrix, but use superscripts for stochastic processes to avoid double indices.

A Multivariate Stochastic Volatility Model of OU Type

Multivariate SV model of OU type

Model the price processes $e^Y := (e^{Y^1}, \dots, e^{Y^d})$ of d assets jointly by

$$\begin{aligned} dY_t &= (\mu + \beta(\Sigma_t))\,dt + \Sigma_t^{\frac{1}{2}}\,dW_t + \rho(dL_t), \quad Y_0 \in \mathbb{R}^d \\ d\Sigma_t &= (A\Sigma_t + \Sigma_t A^{\mathrm{T}})\,dt + dL_t, \quad \Sigma_0 \in \mathbb{S}_d^+, \end{aligned}$$

where

- L is a matrix subordinator, i.e. a matrix valued Lévy process such that

$$L_t - L_s \in \mathbb{S}_d^+ \text{ for all } t \geq s,\ s, t \in \mathbb{R}^+,$$

- W is an $\mathbb{R}^d$-valued Wiener process independent of L,
- linear operators $\beta, \rho : M_d(\mathbb{R}) \to \mathbb{R}^d$, $\mu \in \mathbb{R}^d$,
- and a mean reversion matrix $A \in M_d(\mathbb{R})$ which only has eigenvalues with negative real part.

Marginal dynamics of a single asset

- For example, consider $\boldsymbol{\beta}$, $\boldsymbol{\rho}$ and A diagonal, i.e.

$$\boldsymbol{\beta}(X) = \begin{pmatrix} \beta_1 X^{11} \\ \vdots \\ \beta_d X^{dd} \end{pmatrix}, \quad \boldsymbol{\rho}(X) = \begin{pmatrix} \rho_1 X^{11} \\ \vdots \\ \rho_d X^{dd} \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix}$$

for $X \in M_d(\mathbb{R})$, with $\beta_1, \ldots, \beta_d, \rho_1, \ldots, \rho_d \in \mathbb{R}$, $a_1, \ldots, a_d < 0$.

- In this case

$$\begin{aligned} dY_t^1 &\overset{fidi}{=} (\mu_1 + \beta_1 \Sigma_t^{11})dt + \sqrt{\Sigma_t^{11}} dW_t^1 + \rho_1 dL_t^{11} \\ d\Sigma_t^{11} &= 2a_1 \Sigma_t^{11}\, dt + dL_t^{11}. \end{aligned}$$

where $\overset{fidi}{=}$ denotes equality of all finite-dimensional distributions.

$\Rightarrow$ BNS model for Y^1 (in distribution), analogously for $Y^2, \ldots, Y^d$.

Martingale Conditions and Equivalent Martingale Measures

Denote the triplet of the matrix subordinator L by $(\gamma_L, 0, \kappa_L)$.

Theorem (Martingale Conditions)

The discounted price process $(e^{Y_t - rt})_{t\in\mathbb{R}^+}$ *is a martingale if and only if*

$$\int_{\{X\in\mathbb{S}_d^+:||X||>1\}} e^{\rho^i(X)}\,\kappa_L(dX) < \infty, \quad i = 1,\dots,d$$

and

$$\begin{aligned}\beta(X) &= -\frac{1}{2}(X_{11},\dots,X_{dd})^{\mathrm{T}}, \quad X \in M_d(\mathbb{R}),\\ \mu_i &= r - \int_{\mathbb{S}_d^+}(e^{\rho^i(X)} - 1)\,\kappa_L(dX) - \rho^i(\gamma_L) \quad , i = 1,\dots,d,\end{aligned}$$

where $\rho^i(X)$ *denotes the i-th component of* $\rho(X)$*, and* $r > 0$ *the riskless interest rate.*

Equivalent martingale measures

Define the set of EMMs

$$\mathcal{M} := \{Q : Q \sim P, (e^{Y_t - rt})_{t \in \mathbb{R}^+} \text{ is a } Q\text{-martingale } \}.$$

We can characterize all EMMs (if the filtration is generated by W and L).

Under some measure $Q \in \mathcal{M}$

- L may not be a Lévy process.
- W and L may not be independent.

$\rightarrow$ We do not stay in the same model!

$\rightarrow$ But there exists a subclass that preserves the structure of our model.

Structure preserving EMMs

Consider a finite time interval $[0,T]$. Suppose for simplicity $\gamma_L = 0$. Let $y : \mathbb{S}_d^+ \to \mathbb{R}^{++}$ be such that

1. $\int_{\mathbb{S}_d^+} (\sqrt{y(X)} - 1)^2 \, \kappa_L(dX) < \infty$,
2. $\int_{\{X \in \mathbb{S}_d^+ : \|X\| > 1\}} e^{\rho^i(X)} \, \kappa_L^y(dX) < \infty \quad$ *for all* $i = 1, \dots, d$,

where $\kappa_L^y(B) := \int_B y(X) \, \kappa_L(dX)$ for $B \in \mathcal{B}(\mathbb{S}_d^+)$. Define

$$\psi_t := -\Sigma_t^{-\frac{1}{2}} \left(\mu + \beta(\Sigma_t) + \frac{1}{2} \begin{pmatrix} \Sigma_t^{11} \\ \vdots \\ \Sigma_t^{dd} \end{pmatrix} + \begin{pmatrix} \int_{\mathbb{S}_d^+} (e^{\rho^1(X)} - 1) \, \kappa_L^y(dX) \\ \vdots \\ \int_{\mathbb{S}_d^+} (e^{\rho^d(X)} - 1) \, \kappa_L^y(dX) \end{pmatrix} - \mathbf{1} r \right),$$

where $\mathbf{1} = (1, \dots, 1)^{\mathrm{T}} \in \mathbb{R}^d$.

Then

- $Z = \mathcal{E}\left(\int_0^\cdot \psi_s \, dW_s + (y-1) * (\mu^L - \nu^L)\right)$ is a density process,
- the probability measure Q defined by $\frac{dQ}{dP} = Z_T$ is an EMM,
- $W^Q := W - \int_0^\cdot \psi_s \, ds$ is a Q-standard Brownian motion,
- L is a Q-matrix subordinator with Lévy-Khintchine triplet $\left(0, 0, \kappa_L^y\right)$,
- L and W^Q are independent,
- and the dynamics under Q are given by

$$
\begin{aligned}
dY_t^i &= \left(r - \int_{\mathbb{S}_d^+} (e^{\boldsymbol{\rho}^i(X)} - 1) \, \kappa_L^y(dX) - \frac{1}{2}\Sigma_t^{ii}\right) dt + \left(\Sigma_t^{\frac{1}{2}} \, dW_t^Q\right)^i + \boldsymbol{\rho}^i(dL_t), \\
d\Sigma_t &= (A\Sigma_t + \Sigma_t A^{\mathrm{T}}) \, dt + dL_t.
\end{aligned}
$$

for all $i = 1, \dots, d$.

Characteristic Function and Option Pricing

A heuristic motivation of Fourier Pricing

- Payoff: $f(Y_T)$, for example a basket option with $f(Y_T) = (w_1 e^{Y_T^1} + \ldots + w_d e^{Y_T^d} - K)^+$
- Fourier inversion (for suitable $R \in \mathbb{R}^d$):

$$f(Y_T) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i(iR-u)^{\mathrm{T}} Y_T} \widehat{f}(iR-u) du$$

- Interchanging expectation with the integral above yields the Fourier Pricing Formula:

$$E_Q(e^{-rT} f(Y_T)) = \frac{e^{-rT}}{(2\pi)^d} \int_{\mathbb{R}^d} \Phi_{Y_T}(R+iu) \widehat{f}(iR-u) du \qquad (1)$$

where $\Phi_{Y_T}(R+iu) = E_Q(e^{(R+iu)^{\mathrm{T}} Y_T})$.
$\Rightarrow$ Formula for Φ_{Y_T}? When does (1) hold rigorously? Conditions on Φ_{Y_T}?

Joint characteristic function of (Y_t, Σ_t)

$$E\left[e^{i\langle (y,z),(Y_t,\Sigma_t)\rangle}\right]$$
$$= \exp\left(i y^{\mathrm{T}}(Y_0 + \mu t) + i \mathrm{tr}(\Sigma_0 h_{y,z}(t)) + \int_0^t \Psi_L\left(h_{y,z}(s) + \boldsymbol{\rho}^*(y)\right)\, ds \right),$$

where

$$h_{y,z}(s) = e^{A^{\mathrm{T}} s}\left(z + \mathbf{A}^{-*}\left(\boldsymbol{\beta}^*(y) + \frac{i}{2} y y^{\mathrm{T}} \right)\right) e^{As} - \mathbf{A}^{-*}\left(\boldsymbol{\beta}^*(y) + \frac{i}{2} y y^{\mathrm{T}} \right)$$

and

- $\langle (y,z),(Y_t,\Sigma_t)\rangle = y^{\mathrm{T}} Y_t + \mathrm{tr}(z^{\mathrm{T}}\Sigma_t)$ for $(y,z) \in \mathbb{R}^d \times M_d(\mathbb{R})$
  ($z = 0 \Rightarrow$ characteristic function of Y_t)
- Ψ_L the characteristic exponent of L, i.e. $E(e^{i\mathrm{tr}(ZL_1)}) = e^{\Psi_L(Z)}$,
- $\boldsymbol{\beta}^*, \boldsymbol{\rho}^* : \mathbb{R}^d \to M_d(\mathbb{R})$ are the adjoints of $\boldsymbol{\beta}, \boldsymbol{\rho}$,
- $\mathbf{A}^{-*}$ is the inverse of $\mathbf{A}^* : M_d(\mathbb{R}) \to M_d(\mathbb{R})$, $X \mapsto A^{\mathrm{T}} X + XA$.

Strip of analyticity and absolute integrability of the mgf

Suppose the matrix subordinator L satisfies

$$\int_{\{X \in \mathbb{S}_d^+ : ||X|| \geq 1\}} e^{\mathrm{tr}(RX)} \kappa_L(dX) < \infty \quad \textit{for all } R \in M_d(\mathbb{R}) \textit{ with } ||R|| < \epsilon$$

for some $\epsilon > 0$. Then the moment generating function (mgf) $\Phi_{Y_t}(y) := E\left(e^{y^{\mathrm{T}} Y_t}\right)$ of Y_t is (complex) analytic in the open strip

$$S_\theta := \{y \in \mathbb{C}^d : ||\mathrm{Re}(y)|| < \theta\},$$

where

$$\begin{aligned} \theta &:= -\frac{||\boldsymbol{\rho}||}{(e^{2||A||t}+1)\,||\mathbf{A}^{-*}||} - ||\boldsymbol{\beta}|| + \sqrt{\Delta} > 0, \\ \Delta &:= \left(\frac{||\boldsymbol{\rho}||}{(e^{2||A||t}+1)\,||\mathbf{A}^{-*}||} + ||\boldsymbol{\beta}||\right)^2 + \frac{2\epsilon}{(e^{2||A||t}+1)\,||\mathbf{A}^{-*}||}. \end{aligned}$$

Additionally, the function $\mathbb{R}^d \to \mathbb{C}$, $u \mapsto \Phi_{Y_t}(R+iu)$ is absolutely integrable for all $R \in \mathbb{R}^d$ with $||R|| < \theta$.

Fourier pricing

- Let Q be some EMM
- Choose some $R \in \mathbb{R}^d$ such that $||R|| < \theta$ (i.e. $R + iu \in S_\theta$)
- Multivariate payoff function $f : \mathbb{R}^d \to \mathbb{R}$ such that $x \mapsto e^{-R^{\mathrm{T}}x} f(x) \in L^1 \cap L^\infty$
- $\widehat{f}$: Fourier transform of f
- Arbitrage-free $t = 0$ price of an option with payoff $f(Y_T)$ at time $T > 0$:

$$E_Q\left(e^{-rT} f(Y_T)\right) = \frac{e^{-rT}}{(2\pi)^d} \int_{\mathbb{R}^d} \Phi_{Y_T}(R + iu) \widehat{f}(iR - u)\, du,$$

(cf. Eberlein, Glau, Papapantoleon [2009], univariate: Carr, Madan [99])

- Examples: Plain call options, power options, basket options, spread options, options on the minimum/maximum,...

Pricing of spread options with zero strike

- Formula above not applicable, because the Fourier transform of a zero-strike spread option does not exist.
- Suppose

$$\Phi_{(Y_T^1, Y_T^2)}(R, 1-R) < \infty \quad \textit{for some } R > 1.$$

Then the $t = 0$ price of a zero-strike spread option with payoff $(e^{Y_T^1} - e^{Y_T^2})^+$ at $T > 0$ is given by

$$E_Q(e^{-rT}(e^{Y_T^1} - e^{Y_T^2})^+) = \frac{e^{-rT}}{2\pi} \int_{\mathbb{R}} \frac{\Phi_{(Y_T^1, Y_T^2)}(R + iu, 1 - R - iu)}{(R + iu)(R + iu - 1)} \, du$$

Calibration of the OU-Wishart model in two dimensions

The OU-Wishart model in $d = 2$

$$
\begin{aligned}
d\begin{pmatrix} Y_t^1 \\ Y_t^2 \end{pmatrix} &= \left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \Sigma_t^{11} \\ \Sigma_t^{22} \end{pmatrix} \right) dt + \begin{pmatrix} \Sigma_t^{11} & \Sigma_t^{12} \\ \Sigma_t^{12} & \Sigma_t^{22} \end{pmatrix}^{\frac{1}{2}} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix} + \begin{pmatrix} \rho_1 \, dL_t^{11} \\ \rho_2 \, dL_t^{22} \end{pmatrix} \\
d\begin{pmatrix} \Sigma_t^{11} & \Sigma_t^{12} \\ \Sigma_t^{12} & \Sigma_t^{22} \end{pmatrix} &= \left(\begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix} + \begin{pmatrix} 2a_1\Sigma_t^{11} & (a_1 + a_2)\Sigma_t^{12} \\ (a_1 + a_2)\Sigma_t^{12} & 2a_2\Sigma_t^{22} \end{pmatrix} \right) dt + d\begin{pmatrix} L_t^{11} & L_t^{12} \\ L_t^{12} & L_t^{22} \end{pmatrix}
\end{aligned}
$$

- Here L is a compound Poisson process with $\mathcal{W}_d(2, \Theta)$ Wishart distributed jumps
  (i.e. the jumps have the same distribution as $\Theta^{\frac{1}{2}} X X^{\mathrm{T}} \Theta^{\frac{1}{2}}$, where X is a 2×2 random matrix with independent standard Normal entries).
- Parameters: $\gamma_1, \gamma_2 \geq 0$, $a_1, a_2 < 0$, $\rho_1, \rho_2 \in \mathbb{R}$, $\Theta \in \mathbb{S}_d^+$.

Option pricing in the OU-Wishart model

- $\mathcal{W}_1(2, \Theta_{11})$ = Exponential dist. with mean $2\Theta_{11}$
  $\Rightarrow$ Γ-OU BNS model at the margins (in distribution).
  $\Rightarrow$ Efficient single-asset option pricing as in 1-dim model, speeds up calibration significantly
- Two-asset options: Using integral transform methods.
  - In case $a_1 = a_2$ (same mean reversion): Closed form expression of the mgf Φ_{Y_T} exists.
  - Otherwise: Numerical evaluation of Φ_{Y_T}.
- OU-Wishart model for $d > 2$: Calibration possible using only two-asset options, thus only involving at most two-dimensional integrals.

Modeling the exchange rates of three currencies

- $e^{Y^1} = S^{\$/€}$: Price of 1 € in \$ (EUR/USD exchange rate).
- $e^{Y^2} = S^{\$/£}$: Price of 1 £ in \$ (GBP/USD exchange rate).
- $r_{\$}, r_{€}, r_{£}$ riskless interest rate in each currency. Martingale conditions:

$$\mu_1 = r_{\$} - r_{€} - \frac{\lambda\rho_1}{\frac{1}{2\Theta_{11}} - \rho_1}, \quad \mu_2 = r_{\$} - r_{£} - \frac{\lambda\rho_2}{\frac{1}{2\Theta_{22}} - \rho_2}$$

- Getting a multi-asset option on $S^{\$/€}$ and $S^{\$/£}$:
  - $S^{£/€}$: Price of 1 € in £ (EUR/GBP exchange rate).
  - Call Option on $S^{£/€}$ has £-payoff: $(S_T^{£/€} - K)^+$, i.e. \$-payoff

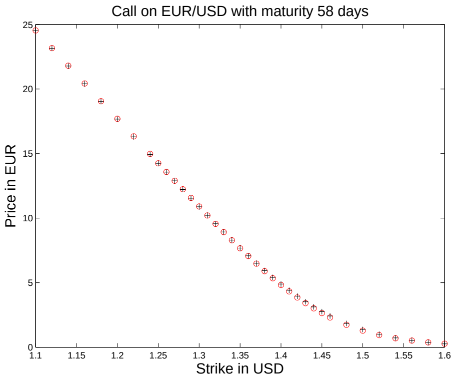
$$S_T^{\$/£}\left(S_T^{£/€} - K\right)^+ = \left(S_T^{\$/€} - KS_T^{\$/£}\right)^+$$

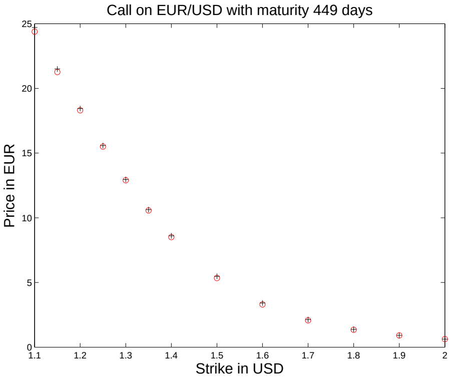
$\Rightarrow$ Zero-strike spread option on $S^{\$/€}$ and $S^{\$/£}$.

Calibration data and results

- Mid-prices of 278 European call options on EUR/USD, GBP/USD and EUR/GBP for different strikes and maturities
- Issued by BNP Paribas and DZ Bank, listed on EUWAX.
- Riskless interest rates: 3 month LIBOR for each currency.
- Minimise the mean squared error (MSE) of market and model prices.
- Runtime: 6 hours.
  RMSE: 0.0586; 0.0683 and 0.0425 at the marginals.
- For comparison: 0.0610 and 0.0320 in unrestricted univariate calibration.
- Assuming same mean reversion $a_1 = a_2$:
  Runtime: 2 hours.
  RMSE: 0.0591; 0.0686 and 0.0439 at the margins.

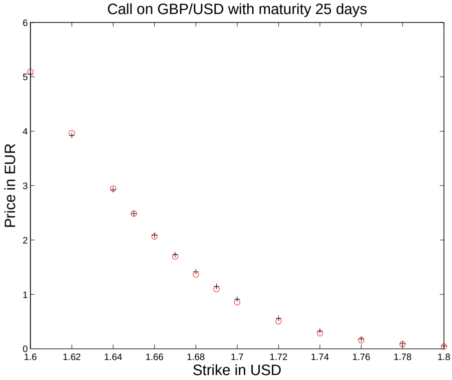
Fit to EUR/USD Call prices

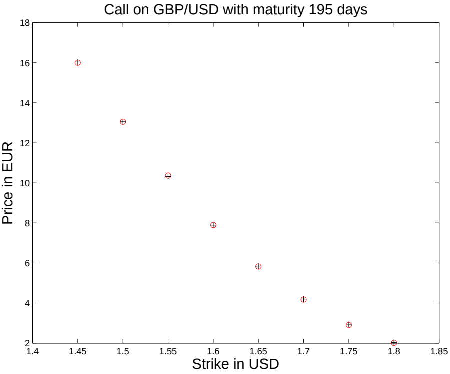




market prices = 'O', model prices = '+'

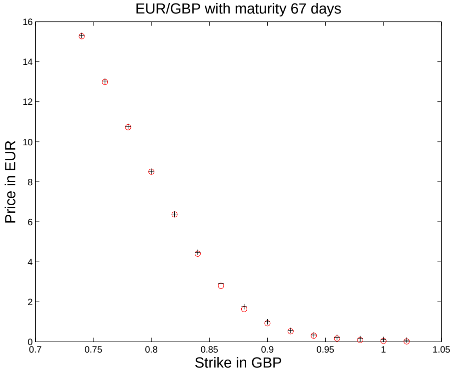
Fit to GBP/USD Call prices

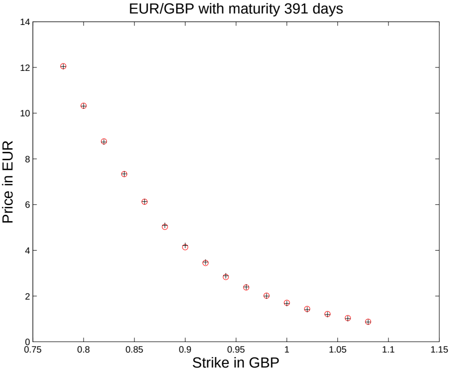




market prices = 'O', model prices = '+'

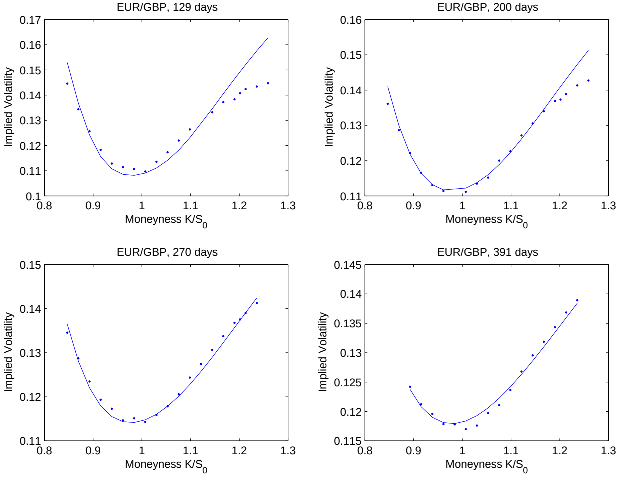
Fit to EUR/GBP Call prices





$\Rightarrow$ Good fit across different strikes and maturities

Implied Volatility Plots



market prices = dot, model prices = solid line

Pricing of Covariance Swaps

- Nontrivial correlation structure $\Rightarrow$ consider covariance swaps, i.e. contracts with payoff $[Y^1, Y^2]_T - K$ with covariance swap rate $K = E([Y^1, Y^2]_T)$.
- Closed form expression for the covariance swap rate in the OU-Wishart model:

$$K = \frac{1}{a_1 + a_2}\left[\left(e^{(a_1+a_2)T} - 1\right)\left(\Sigma_0^{12} + \frac{\lambda n \Theta_{12}}{a_1 + a_2}\right) - T\lambda n \Theta_{12}\right] + T\rho_1\rho_2\lambda n\left(2\Theta_{12}^2 + n\Theta_{11}\Theta_{22}\right)$$

Thank you!

References

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Analysis of Fourier transform valuation formulas and applications. To appear in Applied Mathematical Finance.

Calibration: Resulting parameters

- Different mean reversion

λ	a_1	ρ_1	Θ^{11}	Σ_0^{11}	γ_1	a_2
0.415	-0.313	1.538	0.012	0.017	0	-0.606
ρ_2	Θ^{22}	Σ_0^{22}	γ_2	Θ^{12}	Σ_0^{12}	
-0.211	0.036	0.016	0.003	0.017	0.012	

- Same mean reversion

λ	a	ρ_1	Θ^{11}	Σ_0^{11}	γ_1
0.369	-0.405	1.388	0.014	0.018	0.003
ρ_2	Θ^{22}	Σ_0^{22}	γ_2	Θ^{12}	Σ_0^{12}
-0.274	0.033	0.015	0	0.017	0.012

Closed form expression for Φ_{Y_t} in the OU-Wishart model

$$
\begin{aligned}
E[e^{y^{\mathrm{T}} Y_t}] \;=\; \exp\Bigg\{ & y^{\mathrm{T}}(Y_0 + \mu t) + \frac{e^{2at} - 1}{2a} \operatorname{tr}\left(\Sigma_0 \left(\boldsymbol{\beta}^*(y) + \frac{1}{2} y y^{\mathrm{T}}\right)\right) \\
& + \frac{1}{4a^2}(e^{2at} - 1) \operatorname{tr}\left(\gamma_L \left(\boldsymbol{\beta}^*(y) + \frac{1}{2} y y^{\mathrm{T}}\right)\right) \\
& + t \operatorname{tr}\left(\gamma_L \left(\boldsymbol{\rho}^*(y) - \frac{1}{2a}\left(\boldsymbol{\beta}^*(y) + \frac{1}{2} y y^{\mathrm{T}}\right)\right)\right) \\
& + \frac{\lambda}{2ab_0}\left[\frac{b_1}{d}\left(\arctan\left(\frac{2b_2 + b_1}{d}\right)\right.\right. \\
& \left.- \arctan\left(\frac{2b_2 e^{2at} + b_1}{d}\right)\right) \\
& \left. + \frac{1}{2}\ln\left(\frac{b_0 + b_1 + b_2}{b_2 e^{4at} + b_1 e^{2at} + b_0}\right)\right] + \frac{\lambda}{b_0} t - \lambda t \Bigg\},
\end{aligned}
$$

(b_0, b_1, b_2, d depend on y)

Closed form expression for Φ_{Y_t} in the OU-Wishart model

where

$$\begin{aligned} a &:= a_1 = a_2 \\ b_0 &:= 1 + 4\det(B - C) + 2\mathrm{tr}(B - C) \\ b_1 &:= -8\det(B) + 4\mathrm{tr}(B)\mathrm{tr}(C) - 4\mathrm{tr}(BC) - 2\mathrm{tr}(B) \\ b_2 &:= 4\det(B) \\ d &:= \sqrt{4b_0 b_2 - b_1^2}, \end{aligned}$$

and

$$\begin{aligned} B &:= \frac{1}{2a}\left(\beta^*(y) + \frac{1}{2}yy^{\mathrm{T}}\right)\Theta \\ C &:= \rho^*(y)\Theta. \end{aligned}$$

$\Rightarrow$ Formula is easy to implement.

$\Rightarrow$ Option prices can be computed numerically efficient.