Optimal Control and Filtering

KTH, March, 22, 2010

The exam should be returned by Friday March 26.
The only allowed format is Latex, preferably in pdf form. Handwritten exams
will not be considered. If you do not have a pdf converter in your Latex package,
then please send both the dvi-file and the original Latex file (including macros).
All notation should be clearly defined.
Arguments should be complete and careful.

1. Consider a scalar state process dynamics of the form

$$dX_t = \mu\left(t, X_t, u_t\right) dt + \sigma\left(t, X_t, u_t\right) dW_t$$

   where the scalar control u is without restriction (apart form being adapted).
   Our entire derivation of the HJB equation has so far been based on the
   fact that the objective function is of the form

$$\int_0^T F(t, X_t, u_t) dt + \Phi(X_T).$$

   Sometimes it is natural to consider other criteria, like the exponential
   criterion

$$E\left[e^{\int_0^T F(t, X_t, u_t) dt + \Phi(X_T)}\right]$$

   For this case we define the optimal value function $V(t, x)$ as the supremum
   of

$$E_{t,x}\left[e^{\int_t^T F(s, X_s, u_s) dt + \Phi(X_T)}\right]$$

   Use the ideas of Dynamic Programming in order to show that the HJB
   equation for the expected exponential utility criterion is given by

$$\begin{cases} \dfrac{\partial V}{\partial t}(t, x) + \sup_{u \in R}\left\{V(t, x)F(t, x, u) + \mathcal{A}^u V(t, x)\right\} & = & 0, \\ \\ V(T, x) & = & e^{\Phi(x)}. \end{cases}$$

2. Consider a market model with stock price S and bank account B with
   dynamics given by

$$\begin{aligned} dS_t & = & S_t\mu(Y_t)dt + S_t\sigma(Y_t)dW_t^1, \\ dB_t & = & rB_tdt \end{aligned}$$

Here, the volatility σ and local rate of return μ depend on a "Hidden Markov process" Y with dynamics

$$dY_t = a(Y_t)dt + b(Y_t)dW_t^2.$$

The Wiener processes W^1 and W^2 are assumed to be independent. Analyze, as far as you can, the problem of maximizing logarithmic utility of final wealth

$$E\left[\ln(X_T)\right],$$

where X denotes portfolio value. μ, σ, a and b are given deterministic functions. You are allowed to observe both S and Y so this is not a filtering problem.

3. Consider the filtering model

$$\begin{aligned} dX_t &= a_t dt + dV_t \\ dZ_t &= b_t dt + \sigma_t dW_t \end{aligned}$$

where

- The process σ is $\mathcal{F}_t^Z$ adapted and positive.
- W and V are, possibly correlated, Wiener processes.

Prove, along the lines in the lecture notes, that the filtering equations are given by

$$\begin{aligned} d\widehat{X}_t &= \widehat{a}_t dt + \left[\widehat{D}_t + \frac{1}{\sigma_t}\left\{\widehat{X_t b_t} - \widehat{X}_t \widehat{b}_t\right\}\right] d\nu_t \\ d\nu_t &= \frac{1}{\sigma_t}\left\{dZ_t - \widehat{b}_t dt\right\} \\ D_t &= \frac{d\langle V, W\rangle_t}{dt} \end{aligned}$$

Note: There is a typo in the overhead slides for this formula.

4. Consider a filtered probability space $\left(\Omega, \mathcal{F}, P^0, \mathbf{F}\right)$ carrying an $\mathbf{F}$-Wiener process Z. For any $\mathbf{F}$-adapted process g we can define the process L_t by

$$\begin{aligned} dL_t &= L_t g_t dZ_t \\ L_0 &= 1 \end{aligned}$$

and to emphasize the dependence of g we write $L_t = L_t(g)$. Assume now that we have a process h which is $\mathbf{F}$ adapted and independent of Z. We can then define a new measure P^1 by

$$L_t(h) = \frac{dP^1}{dP^0},$$

on $\mathcal{F}_t$. From Girsanov we then have

$$dZ_t = h_t dt + dW_t,$$

where W is P^1-Wiener. In connection with statistical testing of P^0 against P^1, based on observations of Z, it is natural to use the estimate

$$\widehat{L}_t(h) = E^0 \left[L_t(h) | \mathcal{F}_t^Z \right]$$

as a test statistic, where E^0 denotes integration w.r.t. P^0.

Now define the process $\Pi_t [h]$ by

$$\Pi_t [h] = E^1 \left[h_t | \mathcal{F}_t^Z \right]$$

where the integration is w.r.t. P^1.

Your task is to prove the formula

$$\widehat{L}_t(h) = L_t \left(\Pi_t [h] \right).$$

In other words, prove that

$$d\widehat{L}_t(h) = \widehat{L}_t(h)\Pi_t [h] \, dZ_t.$$

Note: This is a famous formula, referred to as "the separation formula for filtering and detection".

Good Luck!