# Stochastic Optimal Control with Finance Applications

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# Contents

- Dynamic programming.
- Investment theory.
- The martingale approach.
- Filtering theory.
- Optimal investment with partial information.

# 1. Dynamic Programming

- The basic idea.
- Deriving the HJB equation.
- The verification theorem.
- The linear quadratic regulator.

# Problem Formulation

$$\max_{u} \quad E\left[\int_0^T F(t, X_t, u_t)dt + \Phi(X_T)\right]$$

subject to

$$\begin{aligned} dX_t &= \mu\left(t, X_t, u_t\right)dt + \sigma\left(t, X_t, u_t\right)dW_t \\ X_0 &= x_0, \\ u_t &\in U(t, X_t), \quad \forall t. \end{aligned}$$

We will only consider **feedback control laws**, i.e. controls of the form

$$u_t = \mathbf{u}(t, X_t)$$

Terminology:

$$\begin{aligned} X &= \text{state variable} \\ u &= \text{control variable} \\ U &= \text{control constraint} \end{aligned}$$

**Note:** No state space constraints.

# Main idea

- Embedd the problem above in a family of problems indexed by starting point in time and space.

- Tie all these problems together by a PDE–the Hamilton Jacobi Bellman equation.

- The control problem is reduced to the problem of solving the deterministic HJB equation.

# Some notation

- For any fixed vector $u \in R^k$, the functions $\mu^u$, $\sigma^u$ and $C^u$ are defined by

$$\begin{aligned}
\mu^u(t,x) &= \mu(t,x,u), \\
\sigma^u(t,x) &= \sigma(t,x,u), \\
C^u(t,x) &= \sigma(t,x,u)\sigma(t,x,u)'.
\end{aligned}$$

- For any control law $\mathbf{u}$, the functions $\mu^{\mathbf{u}}$, $\sigma^{\mathbf{u}}$, $C^{\mathbf{u}}(t,x)$ and $F^{\mathbf{u}}(t,x)$ are defined by

$$\begin{aligned}
\mu^{\mathbf{u}}(t,x) &= \mu(t,x,\mathbf{u}(t,x)), \\
\sigma^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u}(t,x)), \\
C^{\mathbf{u}}(t,x) &= \sigma(t,x,\mathbf{u}(t,x))\sigma(t,x,\mathbf{u}(t,x))', \\
F^{\mathbf{u}}(t,x) &= F(t,x,\mathbf{u}(t,x)).
\end{aligned}$$

# More notation

- For any fixed vector $u \in R^k$, the partial differential operator $\mathcal{A}^u$ is defined by

$$\mathcal{A}^u = \sum_{i=1}^{n} \mu_i^u(t,x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} C_{ij}^u(t,x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

- For any control law $\mathbf{u}$, the partial differential operator $\mathcal{A}^{\mathbf{u}}$ is defined by

$$\mathcal{A}^{\mathbf{u}} = \sum_{i=1}^{n} \mu_i^{\mathbf{u}}(t,x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} C_{ij}^{\mathbf{u}}(t,x) \frac{\partial^2}{\partial x_i \partial x_j}.$$

- For any control law $\mathbf{u}$, the process $X^{\mathbf{u}}$ is the solution of the SDE

$$dX_t^{\mathbf{u}} = \mu\left(t, X_t^{\mathbf{u}}, \mathbf{u}_t\right) dt + \sigma\left(t, X_t^{\mathbf{u}}, \mathbf{u}_t\right) dW_t,$$

where

$$\mathbf{u}_t = \mathbf{u}(t, X_t^{\mathbf{u}})$$

# Embedding the problem

For every fixed $(t, x)$ the control problem $\mathcal{P}(t, x)$ is defined as the problem to maximize

$$E_{t,x}\left[\int_t^T F(s, X_s^{\mathbf{u}}, u_s)ds + \Phi\left(X_T^{\mathbf{u}}\right)\right],$$

given the dynamics

$$\begin{aligned} dX_s^{\mathbf{u}} &= \mu\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right)ds + \sigma\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right)dW_s, \\ X_t &= x, \end{aligned}$$

and the constraints

$$\mathbf{u}(s, y) \in U, \quad \forall (s, y) \in [t, T] \times R^n.$$

The original problem was $\mathcal{P}(0, x_0)$.

# The optimal value function

- The **value function**

$$\mathcal{J} : R_+ \times R^n \times \mathcal{U} \to R$$

is defined by

$$\mathcal{J}(t, x, \mathbf{u}) = E\left[\int_t^T F(s, X_s^{\mathbf{u}}, \mathbf{u}_s) ds + \Phi\left(X_T^{\mathbf{u}}\right)\right]$$

given the dynamics above.

- The **optimal value function**

$$V : R_+ \times R^n \to R$$

is defined by

$$V(t, x) = \sup_{\mathbf{u} \in \mathcal{U}} \mathcal{J}(t, x, \mathbf{u}).$$

- We want to derive a PDE for $V$.

# Assumptions

We assume:

- There exists an optimal control law $\hat{\mathbf{u}}$.
- The optimal value function $V$ is regular in the sense that $V \in C^{1,2}$.
- A number of limiting procedures in the following arguments can be justified.

# The Bellman Optimality Principle

Dynamic programming relies heavily on the following basic result.

**Proposition:** If $\hat{\mathbf{u}}$ is optimal on the time interval $[t, T]$ then it is also optimal on every subinterval $[s, T]$ with $t \le s \le T$.

**Proof:** Iterated expectations.

# Basic strategy

To derive the PDE do as follows:

- Fix $(t, x) \in (0, T) \times R^n$.
- Choose a real number $h$ (interpreted as a "small" time increment).
- Choose an arbitrary control law $\mathbf{u}$.

Now define the control law $\mathbf{u}^\star$ by

$$\mathbf{u}^\star(s, y) = \begin{cases} \mathbf{u}(s, y), & (s, y) \in [t, t+h] \times R^n \\ \hat{\mathbf{u}}(s, y), & (s, y) \in (t+h, T] \times R^n. \end{cases}$$

In other words, if we use $\mathbf{u}^\star$ then we use the arbitrary control $\mathbf{u}$ during the time interval $[t, t+h]$, and then we switch to the optimal control law during the rest of the time period.

# Basic idea

The whole idea of DynP boils down to the following procedure.

- Given the point $(t, x)$ above, we consider the following two strategies over the time interval $[t, T]$:

  **I:** Use the optimal law $\hat{\mathbf{u}}$.
  **II:** Use the control law $\mathbf{u}^\star$ defined above.

- Compute the expected utilities obtained by the respective strategies.

- Using the obvious fact that Strategy I is least as good as Strategy II, and letting $h$ tend to zero, we obtain our fundamental PDE.

# Strategy values

**Expected utility for strategy I:**

$$\mathcal{J}(t, x, \hat{\mathbf{u}}) = V(t, x)$$

**Expected utility for strategy II:**

- The expected utility for $[t, t+h)$ is given by

$$E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds\right].$$

- Conditional expected utility over $[t+h, T]$, given $(t, x)$:

$$E_{t,x}\left[V(t+h, X_{t+h}^{\mathbf{u}})\right].$$

- Total expected utility for Strategy II is

$$E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds + V(t+h, X_{t+h}^{\mathbf{u}})\right].$$

# Comparing strategies

We have trivially

$$V(t,x) \geq E_{t,x}\left[\int_t^{t+h} F\left(s, X_s^{\mathbf{u}}, \mathbf{u}_s\right) ds + V(t+h, X_{t+h}^{\mathbf{u}})\right].$$

**Remark**
We have equality above if and only if the control law $\mathbf{u}$ is an optimal law $\hat{\mathbf{u}}$.

Now use Itô to obtain

$$V(t+h, X_{t+h}^{\mathbf{u}}) = V(t,x)$$

$$+ \int_t^{t+h} \left\{ \frac{\partial V}{\partial t}(s, X_s^{\mathbf{u}}) + \mathcal{A}^{\mathbf{u}} V(s, X_s^{\mathbf{u}}) \right\} ds$$

$$+ \int_t^{t+h} \nabla_x V(s, X_s^{\mathbf{u}}) \sigma^{\mathbf{u}} dW_s,$$

and plug into the formula above.

We obtain

$$E_{t,x}\left[\int_t^{t+h}\left[F\left(s,X_s^{\mathbf{u}},\mathbf{u}_s\right)+\frac{\partial V}{\partial t}(s,X_s^{\mathbf{u}})+\mathcal{A}^{\mathbf{u}}V\left(s,X_s^{\mathbf{u}}\right)\right]ds\right]\leq 0.$$

**Going to the limit:**
Divide by $h$, move $h$ within the expectation and let $h$ tend to zero.
We get

$$F(t,x,u)+\frac{\partial V}{\partial t}(t,x)+\mathcal{A}^{u}V(t,x)\leq 0,$$

Recall

$$F(t,x,u) + \frac{\partial V}{\partial t}(t,x) + \mathcal{A}^u V(t,x) \leq 0,$$

This holds for all $u = \mathbf{u}(t,x)$, with equality if and only if $\mathbf{u} = \hat{\mathbf{u}}$.

We thus obtain the **HJB equation**

$$\frac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \left\{ F(t,x,u) + \mathcal{A}^u V(t,x) \right\} = 0.$$

# The HJB equation

**Theorem:**
Under suitable regularity assumptions the follwing hold:

I: $V$ satisfies the Hamilton–Jacobi–Bellman equation

$$\begin{aligned} \frac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u V(t,x)\} &= 0, \\ V(T,x) &= \Phi(x), \end{aligned}$$

II: For each $(t,x) \in [0,T] \times R^n$ the supremum in the HJB equation above is attained by $u = \hat{\mathbf{u}}(t,x)$.

# Logic and problem

**Note:** We have shown that **if** $V$ is the optimal value function, and **if** $V$ is regular enough, **then** $V$ satisfies the HJB equation. The HJB eqn is thus derived as a **necessary** condition, and requires strong *ad hoc* regularity assumptions.

**Problem:** Suppose we have solved the HJB equation. Have we then found the optimal value function and the optimal control law?

**Answer:** Yes! This follows from the **Verification tehorem**.

# The Verification Theorem

Suppose that we have two functions $H(t, x)$ and $g(t, x)$, such that

- $H$ is sufficiently integrable, and solves the HJB equation

$$\begin{cases} \dfrac{\partial H}{\partial t}(t,x) + \sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u H(t,x)\} &= 0, \\ H(T,x) &= \Phi(x), \end{cases}$$

- For each fixed $(t, x)$, the supremum in the expression

$$\sup_{u \in U} \{F(t,x,u) + \mathcal{A}^u H(t,x)\}$$

  is attained by the choice $u = g(t, x)$.

Then the following hold.

1. The optimal value function $V$ to the control problem is given by
$$V(t,x) = H(t,x).$$
2. There exists an optimal control law $\hat{\mathbf{u}}$, and in fact
$$\hat{\mathbf{u}}(t,x) = g(t,x)$$

.

# Handling the HJB equation

1. Consider the HJB equation for $V$.
2. Fix $(t, x) \in [0, T] \times R^n$ and solve, the static optimization problem
$$\max_{u \in U} \left[ F(t, x, u) + \mathcal{A}^u V(t, x) \right].$$
Here $u$ is the only variable, whereas $t$ and $x$ are fixed parameters. The functions $F$, $\mu$, $\sigma$ and $V$ are considered as given.
3. The optimal $\hat{u}$, will depend on $t$ and $x$, and on the function $V$ and its partial derivatives. We thus write $\hat{u}$ as
$$\hat{\mathbf{u}} = \hat{\mathbf{u}}(t, x; V). \qquad (1)$$
4. The function $\hat{\mathbf{u}}(t, x; V)$ is our candidate for the optimal control law, but since we do not know $V$ this description is incomplete. Therefore we substitute the expression for $\hat{u}$ into the PDE , giving us the PDE
$$\begin{aligned} \frac{\partial V}{\partial t}(t, x) + F^{\hat{\mathbf{u}}}(t, x) + \mathcal{A}^{\hat{\mathbf{u}}}(t, x) V(t, x) &= 0, \\ V(T, x) &= \Phi(x). \end{aligned}$$
5. Now we solve the PDE above! Then we put the solution $V$ into expression (1). Using the verification theorem we can identify $V$ as the optimal value function, and $\hat{u}$ as the optimal control law.

# Making an Ansatz

- The hard work of dynamic programming consists in solving the highly nonlinear HJB equation

- There are no general analytic methods available for this, so the number of known optimal control problems with an analytic solution is very small indeed.

- In an actual case one usually tries to **guess** a solution, i.e. we typically make a parameterized **Ansatz** for $V$ then use the PDE in order to identify the parameters.

- **Hint:** $V$ often inherits some structural properties from the boundary function $\Phi$ as well as from the instantaneous utility function $F$.

- Most of the known solved control problems have, to some extent, been "rigged" in order to be analytically solvable.

# The Linear Quadratic Regulator

$$\min_{u \in R^k} \quad E\left[\int_0^T \{X_t' Q X_t + u_t' R u_t\} \, dt + X_T' H X_T\right],$$

with dynamics

$$dX_t = \{AX_t + Bu_t\} \, dt + C dW_t.$$

We want to control a vehicle in such a way that it stays close to the origin (the terms $x'Qx$ and $x'Hx$) while at the same time keeping the "energy" $u'Ru$ small.

Here $X_t \in R^n$ and $\mathbf{u}_t \in R^k$, and we impose no control constraints on $u$.

The matrices $Q$, $R$, $H$, $A$, $B$ and $C$ are assumed to be known. We may WLOG assume that $Q$, $R$ and $H$ are symmetric, and we assume that $R$ is positive definite (and thus invertible).

# Handling the Problem

The HJB equation becomes

$$
\left\{
\begin{array}{rcl}
\dfrac{\partial V}{\partial t}(t,x) & + & \inf_{u \in R^k} \left\{ x'Qx + u'Ru + [\nabla_x V](t,x)\,[Ax+Bu] \right\} \\
 & + & \frac{1}{2}\sum_{i,j} \frac{\partial^2 V}{\partial x_i \partial x_j}(t,x)\,[CC']_{i,j} = 0, \\
V(T,x) & = & x'Hx.
\end{array}
\right.
$$

For each fixed choice of $(t,x)$ we now have to solve the static unconstrained optimization problem to minimize

$$
u'Ru + [\nabla_x V](t,x)\,[Ax+Bu]\,.
$$

The problem was:

$$\min_{u} \quad u'Ru + [\nabla_x V](t,x)\,[Ax + Bu]\,.$$

Since $R > 0$ we set the gradient to zero and obtain

$$2u'R = -(\nabla_x V)B,$$

which gives us the optimal $u$ as

$$\hat{u} = -\frac{1}{2}R^{-1}B'(\nabla_x V)'.$$

**Note:** This is our candidate of optimal control law, but it depends on the unkown function $V$.

We now make an educated guess about the shape of $V$.

From the boundary function $x'Hx$ and the term $x'Qx$ in the cost function we make the Ansatz

$$V(t,x) = x'P(t)x + q(t),$$

where $P(t)$ is a symmetric matrix function, and $q(t)$ is a scalar function.

With this trial solution we have,

$$\begin{aligned}
\frac{\partial V}{\partial t}(t,x) &= x'\dot{P}x + \dot{q}, \\
\nabla_x V(t,x) &= 2x'P, \\
\nabla_{xx} V(t,x) &= 2P \\
\hat{u} &= -R^{-1}B'Px.
\end{aligned}$$

Inserting these expressions into the HJB equation we get

$$\begin{aligned}
& x'\left\{\dot{P} + Q - PBR^{-1}B'P + A'P + PA\right\}x \\
& +\dot{q} + tr[C'PC] = 0.
\end{aligned}$$

We thus get the following matrix ODE for $P$

$$\begin{cases} \dot{P} = PBR^{-1}B'P - A'P - PA - Q, \\ P(T) = H. \end{cases}$$

and we can integrate directly for $q$:

$$\begin{cases} \dot{q} = -tr[C'PC], \\ q(T) = 0. \end{cases}$$

The matrix equation is a **Riccati equation**. The equation for $q$ can then be integrated directly.

**Final Result for LQ:**

$$\begin{aligned} V(t,x) &= x'P(t)x + \int_t^T tr[C'P(s)C]ds, \\ \hat{\mathbf{u}}(t,x) &= -R^{-1}B'P(t)x. \end{aligned}$$

# 2. Portfolio Theory

- Problem formulation.
- An extension of HJB.
- The simplest consutmption-investment problem.
- The Merton fund separation results.

# Recap of Basic Facts

We consider a market with $n$ assets.

$$\begin{aligned}
S_t^i &= \text{price of asset No } i, \\
h_t^i &= \text{units of asset No } i \text{ in portfolio} \\
w_t^i &= \text{portfolio weight on asset No } i \\
X_t &= \text{portfolio value} \\
c_t &= \text{consumption rate}
\end{aligned}$$

We have the relations

$$X_t = \sum_{i=1}^{n} h_t^i S_t^i, \quad w_t^i = \frac{h_t^i S_t^i}{X_t}, \quad \sum_{i=1}^{n} u_t^i = 1.$$

**Basic equation:**
Dynamics of self financing portfolio in terms of relative weights

$$dX_t = X_t \sum_{i=1}^{n} w_t^i \frac{dS_t^i}{S_t^i} - c_t dt$$

# Simplest model

Assume a scalar risky asset and a constant short rate.

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t \\
dB_t &= r B_t dt
\end{aligned}
$$

We want to maximize expected utility over time

$$
\max_{w^0, w^1, c} \quad E\left[\int_0^T F(t, c_t) dt + \Phi(X_T)\right]
$$

Dynamics

$$
dX_t = X_t \left[u_t^0 r + w_t^1 \alpha\right] dt - c_t dt + w_t^1 \sigma X_t dW_t,
$$

Constraints

$$
\begin{aligned}
c_t &\geq 0, \ \forall t \geq 0, \\
w_t^0 + w_t^1 &= 1, \ \forall t \geq 0.
\end{aligned}
$$

# Nonsense!

# What are the problems?

- We can obtain umlimited uttility by simply consuming arbitrary large amounts.
- The wealth will go negative, but there is nothing in the problem formulations which prohibits this.
- We would like to impose a constratin of type $X_t \geq 0$ but this is a **state constraint** and DynP does not allow this.

**Good News:**
DynP can be generalized to handle (some) problems of this kind.

# Generalized problem

Let $D$ be a nice open subset of $[0,T]\times R^n$ and consider the following problem.

$$\max_{u\in U} \quad E\left[\int_0^\tau F(s, X_s^{\mathbf{u}}, \mathbf{u}_s)ds + \Phi\left(\tau, X_\tau^{\mathbf{u}}\right)\right].$$

Dynamics:

$$\begin{aligned} dX_t &= \mu\left(t, X_t, u_t\right)dt + \sigma\left(t, X_t, u_t\right)dW_t, \\ X_0 &= x_0, \end{aligned}$$

The **stopping time** $\tau$ is defined by

$$\tau = \inf\left\{t \geq 0 \;|(t, X_t) \in \partial D\right\} \wedge T.$$

# Generalized HJB

**Theorem:** Given enough regularity the follwing hold.

1. The optimal value function satisfies

$$\left\{ \begin{array}{rcll} \dfrac{\partial V}{\partial t}(t,x) + \sup_{u \in U} \left\{ F(t,x,u) + \mathcal{A}^u V(t,x) \right\} & = & 0, & \forall (t,x) \in D \\ V(t,x) & = & \Phi(t,x), & \forall (t,x) \in \partial D. \end{array} \right.$$

2. We have an obvious verification theorem.

# Reformulated problem

$$\max_{c \geq 0,\ w \in R} \quad E\left[\int_0^\tau F(t, c_t)dt + \Phi(X_T)\right]$$

where

$$\tau = \inf\{t \geq 0\ |X_t = 0\} \wedge T.$$

with notation:

$$\begin{aligned} w^1 &= w, \\ w^0 &= 1 - w \end{aligned}$$

Thus no constraint on $w$.

Dynamics

$$dX_t = w_t\left[\alpha - r\right]X_t dt + \left(rX_t - c_t\right)dt + w\sigma X_t dW_t,$$

# HJB Equation

$$\begin{aligned}
\frac{\partial V}{\partial t}+\sup_{c\geq 0, w\in R}\left\{F(t,c)+wx(\alpha-r)\frac{\partial V}{\partial x}+(rx-c)\frac{\partial V}{\partial x}+\frac{1}{2}x^2w^2\sigma^2\frac{\partial^2 V}{\partial x^2}\right\} &= 0,\\
V(T,x) &= 0,\\
V(t,0) &= 0.
\end{aligned}$$

We now specialize (why?) to

$$F(t,c)=e^{-\delta t}c^{\gamma},$$

so we have to maximize

$$e^{-\delta t}c^{\gamma}+wx(\alpha-r)\frac{\partial V}{\partial x}+(rx-c)\frac{\partial V}{\partial x}+\frac{1}{2}x^2w^2\sigma^2\frac{\partial^2 V}{\partial x^2},$$

# Analysis of the HJB Equation

In the embedde static problem we maximize, over $c$ and $w$,

$$e^{-\delta t}c^{\gamma} + wx(\alpha - r)\frac{\partial V}{\partial x} + (rx - c)\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial x^2},$$

First order conditions:

$$\begin{aligned} \gamma c^{\gamma - 1} &= e^{\delta t} V_x, \\ w &= \frac{-V_x}{x \cdot V_{xx}} \cdot \frac{\alpha - r}{\sigma^2}, \end{aligned}$$

**Ansatz:**

$$V(t, x) = e^{-\delta t} h(t) x^{\gamma},$$

Because of the boundary conditions, we must demand that

$$h(T) = 0. \qquad (2)$$

Given a $V$ of this form we have (using $\cdot$ to denote the time derivative)

$$
\begin{aligned}
\frac{\partial V}{\partial t} &= e^{-\delta t}\dot{h}x^{\gamma} - \delta e^{-\delta t} h x^{\gamma}, \\
\frac{\partial V}{\partial x} &= \gamma e^{-\delta t} h x^{\gamma-1}, \\
\frac{\partial^2 V}{\partial x^2} &= \gamma(\gamma-1) e^{-\delta t} h x^{\gamma-2}.
\end{aligned}
$$

giving us

$$
\begin{aligned}
\hat{\mathbf{w}}(t,x) &= \frac{\alpha - r}{\sigma^2(1-\gamma)}, \\
\hat{\mathbf{c}}(t,x) &= x h(t)^{-1/(1-\gamma)}.
\end{aligned}
$$

Plug all this into HJB!

After rearrangements we obtain

$$x^{\gamma}\left\{\dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)}\right\} = 0,$$

where the constants $A$ and $B$ are given by

$$\begin{aligned} A &= \frac{\gamma(\alpha-r)^2}{\sigma^2(1-\gamma)} + r\gamma - \frac{1}{2}\frac{\gamma(\alpha-r)^2}{\sigma^2(1-\gamma)} - \delta \\ B &= 1-\gamma. \end{aligned}$$

If this equation is to hold for all $x$ and all $t$, then we see that $h$ must solve the ODE

$$\begin{aligned} \dot{h}(t) + Ah(t) + Bh(t)^{-\gamma/(1-\gamma)} &= 0, \\ h(T) &= 0. \end{aligned}$$

An equation of this kind is known as a **Bernoulli equation**, and it can be solved explicitly.

We are done.

# Merton's Mutal Fund Theorems

## 1. The case with no risk free asset

We consider $n$ risky assets with dynamics

$$dS_i = S_i \alpha_i dt + S_i \sigma_i dW, \quad i = 1, \dots, n$$

where $W$ is Wiener in $R^k$. On vector form:

$$dS = D(S)\alpha dt + D(S)\sigma dW.$$

where

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \quad \sigma = \begin{bmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{bmatrix}$$

$D(S)$ is the diagonal matrix

$$D(S) = diag[S_1, \dots, S_n].$$

Wealth dynamics

$$dX = Xw'\alpha dt - cdt + Xw'\sigma dW.$$

# Formal problem

$$\max_{c,w} \quad E\left[\int_0^{\tau} F(t, c_t)dt\right]$$

given the dynamics

$$dX = Xw'\alpha dt - cdt + Xw'\sigma dW.$$

and constraints

$$e'w = 1, \quad c \geq 0.$$

**Assumptions:**

- The vector $\alpha$ and the matrix $\sigma$ are constant and deterministic.
- The volatility matrix $\sigma$ has full rank so $\sigma\sigma'$ is positive definite and invertible.

**Note:** $S$ does not turn up in the $X$-dynamics so $V$ is of the form

$$V(t, x, s) = V(t, x)$$

The HJB equation is

$$\left\{\begin{array}{rcl}\frac{\partial V}{\partial t}(t,x)+\sup_{e'w=1,\ c\geq 0}\{F(t,c)+\mathcal{A}^{c,w}V(t,x)\} & = & 0,\\ V(T,x) & = & 0,\\ V(t,0) & = & 0.\end{array}\right.$$

where

$$\mathcal{A}^{c,w}V = xw'\alpha\frac{\partial V}{\partial x} - c\frac{\partial V}{\partial x} + \frac{1}{2}x^2 w'\Sigma w\frac{\partial^2 V}{\partial x^2},$$

and where the matrix $\Sigma$ is given by

$$\Sigma = \sigma\sigma'.$$

The HJB equation is

$$\begin{cases} V_t(t,x) + \sup_{w'e=1,\ c\geq 0} \left\{ F(t,c) + (xw'\alpha - c)V_x(t,x) + \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x) \right\} & = & 0, \\ V(T,x) & = & 0, \\ V(t,0) & = & 0. \end{cases}$$

where $\Sigma = \sigma\sigma'$.

If we relax the constraint $w'e = 1$, the Lagrange function for the static optimization problem is given by

$$L = F(t,c) + (xw'\alpha - c)V_x(t,x) + \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x) + \lambda\left(1 - w'e\right).$$

$$
\begin{aligned}
L &= F(t,c) + (xw'\alpha - c)V_x(t,x) \\
&+ \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x) + \lambda\left(1 - w'e\right).
\end{aligned}
$$

The first order condition for $c$ is

$$
\frac{\partial F}{\partial c}(t,c) = V_x(t,x).
$$

The first order condition for $w$ is

$$
x\alpha' V_x + x^2 V_{xx} w'\Sigma = \lambda e',
$$

so we can solve for $w$ in order to obtain

$$
\hat{w} = \Sigma^{-1}\left[\frac{\lambda}{x^2 V_{xx}} e - \frac{x V_x}{x^2 V_{xx}}\alpha\right].
$$

Using the relation $e'w = 1$ this gives $\lambda$ as

$$
\lambda = \frac{x^2 V_{xx} + x V_x e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e},
$$

Inserting $\lambda$ gives us, after some manipulation,

$$\hat{w} = \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e + \frac{V_x}{xV_{xx}}\Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right].$$

We can write this as

$$\hat{\mathbf{w}}(t) = g + Y(t)h,$$

where the fixed vectors $g$ and $h$ are given by

$$\begin{aligned} g &= \frac{1}{e'\Sigma^{-1}e}\Sigma^{-1}e, \\ h &= \Sigma^{-1}\left[\frac{e'\Sigma^{-1}\alpha}{e'\Sigma^{-1}e}e - \alpha\right], \end{aligned}$$

whereas $Y$ is given by

$$Y(t) = \frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}.$$

We had

$$\hat{\mathbf{w}}(t) = g + Y(t)h,$$

Thus we see that the optimal portfolio is moving stochastically along the one-dimensional "optimal portfolio line"

$$g + sh,$$

in the $(n-1)$-dimensional "portfolio hyperplane" $\Delta$, where

$$\Delta = \{w \in R^n \mid e'w = 1\}.$$

If we fix two points on the optimal portfolio line, say $w^a = g + ah$ and $w^b = g + bh$, then any point $w$ on the line can be written as an affine combination of the basis points $w^a$ and $w^b$. An easy calculation shows that if $w^s = g + sh$ then we can write

$$w^s = \mu w^a + (1-\mu)w^b,$$

where

$$\mu = \frac{s-b}{a-b}.$$

# Mutual Fund Theorem

There exists a family of mutual funds, given by $w^s = g + sh$, such that

1. For each fixed $s$ the portfolio $w^s$ stays fixed over time.

2. For fixed $a, b$ with $a \neq b$ the optimal portfolio $\hat{\mathbf{w}}(t)$ is, obtained by allocating all resources between the fixed funds $w^a$ and $w^b$, i.e.

$$\hat{\mathbf{w}}(t) = \mu^a(t) w^a + \mu^b(t) w^b,$$

3. The relative proportions $(\mu^a, \mu^b)$ of wealth allocated to $w^a$ and $w^b$are given by

$$\begin{aligned} \mu^a(t) &= \frac{Y(t) - b}{a - b}, \\ \mu^b(t) &= \frac{a - Y(t)}{a - b}. \end{aligned}$$

# The case with a risk free asset

Again we consider the standard model

$$dS = D(S)\alpha dt + D(S)\sigma dW(t),$$

We also assume the risk free asset $B$ with dynamics

$$dB = rBdt.$$

We denote $B = S_0$ and consider portfolio weights $(w_0, w_1, \ldots, w_n)'$ where $\sum_0^n w_i = 1$. We then eliminate $w_0$ by the relation

$$w_0 = 1 - \sum_1^n w_i,$$

and use the letter $w$ to denote the portfolio weight vector for the risky assets only. Thus we use the notation

$$w = (w_1, \ldots, w_n)',$$

**Note:** $w \in R^n$ without constraints.

# HJB

We obtain

$$dX = X \cdot w'(\alpha - re)dt + (rX - c)dt + X \cdot w'\sigma dW,$$

where $e = (1, 1, \ldots, 1)'$.

The HJB equation now becomes

$$\begin{cases} V_t(t,x) + \sup_{c \geq 0, w \in R^n} \{F(t,c) + \mathcal{A}^{c,w} V(t,x)\} & = & 0, \\ V(T,x) & = & 0, \\ V(t,0) & = & 0, \end{cases}$$

where

$$\begin{aligned} \mathcal{A}^c V &= xw'(\alpha - re)V_x(t,x) + (rx - c)V_x(t,x) \\ &+ \frac{1}{2}x^2 w'\Sigma w V_{xx}(t,x). \end{aligned}$$

# First order conditions

We maximize

$$F(t,c) + xw'(\alpha - re)V_x + (rx - c)V_x + \frac{1}{2}x^2 w'\Sigma w V_{xx}$$

with $c \geq 0$ and $w \in R^n$.

The first order conditions are

$$\begin{aligned}
\frac{\partial F}{\partial c}(t,c) &= V_x(t,x), \\
\hat{w} &= -\frac{V_x}{xV_{xx}}\Sigma^{-1}(\alpha - re),
\end{aligned}$$

with geometrically obvious economic interpretation.

# Mutual Fund Separation Theorem

1. The optimal portfolio consists of an allocation between two fixed mutual funds $w^0$ and $w^f$.

2. The fund $w^0$ consists only of the risk free asset.

3. The fund $w^f$ consists only of the risky assets, and is given by
$$w^f = \Sigma^{-1}(\alpha - re).$$

4. At each $t$ the optimal relative allocation of wealth between the funds is given by

$$\begin{aligned}
\mu^f(t) &= -\frac{V_x(t, X(t))}{X(t)V_{xx}(t, X(t))}, \\
\mu^0(t) &= 1 - \mu^f(t).
\end{aligned}$$

# 3. The Martingale Approach

- Decoupling the wealth profile from the portfolio choice.
- Lagrange relaxation.
- Solving the general wealth problem.
- Example: Log utility.
- Example: The numeraire portfolio.
- Computing the optimal portfolio.
- The Merton fund separation theorems from a martingale perspective..

# Problem Formulation

Standard model with internal filtration

$$\begin{aligned} dS_t &= D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t, \\ dB_t &= rB_t dt. \end{aligned}$$

**Assumptions:**

- Drift and diffusion terms are allowed to be arbitrary adapted processes.
- The market is **complete**.
- We have a given initial wealth $x_0$

**Problem:**

$$\max_{h \in \mathcal{H}} \quad E^P\left[\Phi(X_T)\right]$$

where

$$\mathcal{H} = \{\text{self financing portfolios}\}$$

given the initial wealth $X_0 = x_0$.

# Some observations

- In a complete market, there is a unique martingale measure $Q$.
- Every claim $Z$ satisfying the budget constraint

$$e^{-rT} E^Q [Z] = x_0,$$

is attainable by an $h \in \mathcal{H}$ and vice versa.

- We can thus write our problem as

$$\max_{Z} \quad E^P [\Phi(Z)]$$

subject to the constraint

$$e^{-rT} E^Q [Z] = x_0.$$

- We can forget the wealth dynamics!

# Basic Ideas

Our problem was

$$\max_{Z} \quad E^P\left[\Phi(Z)\right]$$

subject to

$$e^{-rT}E^Q\left[Z\right] = x_0.$$

**Idea I:**

We can **decouple** the optimal portfolio problem:

- Finding the optimal wealth profile $\hat{Z}$.
- Given $\hat{Z}$, find the replicating portfolio.

**Idea II:**

- Rewrite the constraint under the measure $P$.
- Use Lagrangian techniques to relax the constraint.

# Lagrange formulation

Problem:

$$\max_{Z} \quad E^P\left[\Phi(Z)\right]$$

subject to

$$e^{-rT}E^P\left[L_T Z\right] = x_0.$$

Here $L$ is the likelihood process, i.e.

$$L_T = \frac{dQ}{dP}, \quad \text{on } \mathcal{F}_T$$

The Lagrangian of this is

$$\mathcal{L} = E^P\left[\Phi(Z)\right] + \lambda\left\{x_0 - e^{-rT}E^P\left[L_T Z\right]\right\}$$

i.e.

$$\mathcal{L} = E^P\left[\Phi(Z) - \lambda e^{-rT}L_T Z\right] + \lambda x_0$$

# The optimal wealth profile

Given enough convexity and regularity we now expect, given the dual variable $\lambda$, to find the optimal $Z$ by maximizing

$$\mathcal{L} = E^P \left[ \Phi(Z) - \lambda e^{-rT} L_T Z \right] + \lambda x_0$$

over unconstrained $Z$, i.e. to maximize

$$\int_\Omega \left\{ \Phi(Z(\omega)) - \lambda e^{-rT} L_T(\omega) Z(\omega) \right\} dP(\omega)$$

**This is a trivial problem!**

We can simply maximize $Z(\omega)$ for each $\omega$ separately.

$$\max_z \quad \left\{ \Phi(z) - \lambda e^{-rT} L_T z \right\}$$

# The optimal wealth profile

Our problem:

$$\max_{z} \quad \left\{ \Phi(z) - \lambda e^{-rT} L_T z \right\}$$

First order condition

$$\Phi'(z) = \lambda e^{-rT} L_T$$

The optimal $Z$ is thus given by

$$\hat{Z} = G\left(\lambda e^{-rT} L_T\right)$$

where

$$G(y) = \left[\Phi'\right]^{-1}(y).$$

The dual varaiable $\lambda$ is determined by the constraint

$$e^{-rT} E^P\left[L_T \hat{Z}\right] = x_0.$$

# Example – log utility

Assume that

$$\Phi(x) = \ln(x)$$

Then

$$g(y) = \frac{1}{y}$$

Thus

$$\hat{Z} = G\left(\lambda e^{-rT} L_T\right) = \frac{1}{\lambda} e^{rT} L_T^{-1}$$

Finally $\lambda$ is determined by

$$e^{-rT} E^P \left[ L_T \hat{Z} \right] = x_0.$$

i.e.

$$e^{-rT} E^P \left[ L_T \frac{1}{\lambda} e^{rT} L_T^{-1} \right] = x_0.$$

so $\lambda = x_0^{-1}$ and

$$\hat{Z} = x_0 e^{rT} L_T^{-1}$$

# The Numeraire Portfolio

**Standard approach:**

- Choose a fixed numeraire (portfolio) $N$.
- Find the corresponding martingale measure, i.e. find $Q^N$ s.t.

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are $Q^N$ -martingales.

**Alternative approach:**

- Choose a fixed measure $Q$.
- Find numeraire $N$ such that $Q = Q^N$.

**Special case:**

- Set $Q = P$
- Find numeraire $N$ such that $Q^N = P$ i.e. such that

$$\frac{B}{N}, \quad \text{and} \quad \frac{S}{N}$$

are $Q^N$ -martingales under the **objective** measure $P$.
- This $N$ is the **numeraire portfolio**.

# Log utility and the numeraire portfolio

**Definition:**
The **growth optimal portfolio** (GOP) is the portfolio which is optimal for log utility (for arbitrary terminal date $T$.

**Theorem:**
Assume that $X$ is GOP. Then $X$ is the numeraire portfolio.

**Proof:**
We have to show that the process

$$Y_t = \frac{S_t}{X_t}$$

is a $P$ martingale. From above we know that

$$X_T = x_0 e^{rT} L_T^{-1}.$$

We also have (why?)

$$X_t = e^{-r(T-t)} E^Q \left[ X_T \middle| \mathcal{F}_t \right] = e^{-r(T-t)} E^P \left[ \frac{X_T L_T}{L_t} \middle| \mathcal{F}_t \right]$$

Thus

$$X_t = e^{-r(T-t)} E^P \left[ \left. \frac{X_T L_T}{L_t} \right| \mathcal{F}_t \right]$$
$$= e^{-r(T-t)} E^Q \left[ \left. \frac{x_0 e^{rT} L_T}{L_T L_t} \right| \mathcal{F}_t \right] = x_0 e^{rt} L_t^{-1}.$$

as expected.

Thus

$$\frac{S_t}{X_t} = x_0^{-1} e^{-rt} S_t L_t$$

which is a $P$ martingale, since $x_0^{-1} e^{-rt} S_t$ is a $Q$ martingale.

# Back to general case: Computing $L_T$

We recall

$$\hat{Z} = G\left(\lambda e^{-rT} L_T\right).$$

The likelihood process $L$ is computed by using Girsanov. We recall

$$dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t,$$

We know from Girsanov that

$$dL_t = L_t \varphi_t^{\star} dW_t$$

so

$$dW_t = \varphi_t dt + dW_t^Q$$

where $W^Q$ is $Q$-Wiener.

Thus

$$dS_t = D(S_t)\left\{\alpha_t + \sigma_t \varphi_t\right\} dt + D(S_t)\sigma_t dW_t^Q,$$

# Computing $L_T$, continued

Recall

$$dS_t = D(S_t)\left\{\alpha_t + \sigma_t\varphi_t\right\}dt + D(S_t)\sigma_t dW_t^Q,$$

The kernel $\varphi$ is determined by the martingale measure condition

$$\alpha_t + \sigma_t\varphi_t = \mathbf{r}$$

where

$$\mathbf{r} = \begin{bmatrix} r \\ \vdots \\ r \end{bmatrix}$$

Market completeness implies that $\sigma_t$ is invertible so

$$\varphi_t = \sigma_t^{-1}\left\{\mathbf{r} - \alpha_t\right\}$$

and

$$L_T = \exp\left(\int_0^T \varphi_t dW_t - \frac{1}{2}\int_0^T \|\varphi_t\|^2 dt\right)$$

# Finding the optimal portfolio

- We can easily compute the optimal wealth profile.
- How do we compute the optimal portfolio?

Recall:

$$dS_t = D(S_t)\alpha_t dt + D(S_t)\sigma_t dW_t,$$

wealth dynamics

$$dX_t = h_t^B dB_t + h_t^S dS_t$$

or

$$dX_t = X_t u_t^B r dt + X_t u_t^S D(S_t)^{-1} dS_t$$

where

$$h^S = (h^1, \ldots, h^n), \quad u^S = (u^1, \ldots, u^n)$$

Assume for simplicity that $r = 0$ or consider normalized prices.

Recall wealth dynamics

$$dX_t = h_t^S dS_t$$

alternatively

$$dX_t = h_t^S D(S_t)\sigma_t dW_t^Q$$

alternatively

$$dX_t = X_t u_t^S \sigma_t dW_t^Q$$

Obvious facts:

- $X$ is a $Q$ martingale.
- $X_T = \hat{Z}$

Thus the optimal wealth process is determined by

$$X_t = E^Q\left[\hat{Z}\middle|\mathcal{F}_t\right]$$

Recall

$$X_t = E^Q\left[\hat{Z}\middle|\mathcal{F}_t\right]$$

Martingale representation theorem gives us

$$dX_t = \xi_t dW_t^Q,$$

but also

$$\begin{aligned} dX_t &= X_t u_t^S \sigma_t dW_t^Q \\ dX_t &= h_t^S D(S_t) \sigma_t dW_t^Q \end{aligned}$$

Thus $u^S$ and $h_t^S$ are determined by

$$\begin{aligned} u_t^S &= \frac{1}{X_t} \xi_t \sigma_t^{-1} \\ h_t^S &= \xi_t \sigma_t^{-1} D(S_t)^{-1}. \end{aligned}$$

and

$$u_t^B = 1 - u_t^S \mathbf{e}, \quad h_t^B = X_t - h_t^S S_t$$

# How do we find $\xi$?

Recall

$$X_t = E^Q\left[\hat{Z}\middle|\mathcal{F}_t\right]$$
$$dX_t = \xi_t dW_t^Q,$$

We need to compute $\xi$.

In a Markovian framework this follows direcetly from the Itô formula.

Recall

$$\hat{Z} = H(L_T) = G(\lambda L_T)$$

where

$$G = [\Phi']^{-1}$$

and

$$dL_t = L_t\varphi_t^\star dW_t,$$
$$dW = \varphi dt + dW_t^Q$$

so

$$dL_t = L_t\|\varphi_t\|^2 dt + L_t\varphi_t^\star dW_t^Q$$

# Finding $\xi$

If the model is Markovian we have

$$\alpha_t = \alpha(S_t), \quad \sigma_t = \sigma(S_t), \quad \varphi_t = \sigma(S_t)^{-1}\left\{\alpha(S_t) - \mathbf{r}\right\}$$

so

$$\begin{aligned} X_t &= E^Q\left[H(L_T)\middle|\, \mathcal{F}_t\right] \\ dS_t &= D(S_t)\sigma(S_t)dW_t^Q, \\ dL_t &= L_t\|\varphi(S_t)\|^2 dt + L_t\varphi(S_t)^\star dW_t^Q \end{aligned}$$

Thus we have

$$X_t = F(t, S_t, L_t)$$

where, by the Kolmogorov backward equation

$$\begin{aligned} F_t + L\|\varphi\|^2 F_L + \frac{1}{2}L^2\|\varphi\|^2 F_{LL} + \frac{1}{2}tr\left\{\sigma F_{ss}\sigma^\star\right\} &= 0, \\ F(T, s, L) &= H(L) \end{aligned}$$

# Finding $\xi$, contd.

We had

$$X_t = F(t, S_t, L_t)$$

and Itô gives us

$$dX_t = \{F_S D(S_t)\sigma(S_t) + F_L L_t \varphi^\star(S_t)\} dW_t$$

Thus

$$\xi_t = F_S D(S_t)\sigma(S_t) + F_L L_t \varphi^\star(S_t).$$

and

$$\begin{aligned} u_t^S &= \frac{1}{X_t}\xi_t \sigma(S_t)^{-1} \\ h_t^S &= \xi_t \sigma(S_t)^{-1} D(S_t)^{-1}. \end{aligned}$$

# Mutual Funds – Martingale Version

We now assume constant parameters

$$\alpha(s) = \alpha, \quad \sigma(s) = \sigma, \quad \varphi(s) = \varphi$$

We recall

$$\begin{aligned} X_t &= E^Q \left[ H(L_T) \middle| \mathcal{F}_t \right] \\ dL_t &= L_t \|\varphi\|^2 dt + L_t \varphi^\star dW_t^Q \end{aligned}$$

Now $L$ is Markov so we have (without any $S$)

$$X_t = F(t, L_t)$$

Thus

$$\xi_t = F_L L_t \varphi^\star, \quad u_t^S = \frac{F_L L_t}{X_t} \varphi^\star \sigma^{-1}$$

and we have fund separation with the fixed risky fund given by

$$w = \varphi^\star \sigma^{-1} = \left\{ \mathbf{r}^\star - \alpha^\star \right\} \left\{ \sigma \sigma^\star \right\}^{-1}.$$

# 3. Filtering theory

- Motivational problem.
- The Innovations process.
- The non-linear FKK filtering equations.
- The Wonham filter.
- The Kalman filter.

# An investment problem with stochastic rate of return

**Model:**

$$dS_t = S_t \alpha(Y_t) dt + S_t \sigma dW_t$$

$W$ is scalar and $Y$ is some factor process. We assume that $(S, Y)$ is Markov and adapted to the filtration $\mathbf{F}$.

Wealth dynamics

$$dX_t = X_t \left[ r + u_t \left( \alpha - r \right) \right] dt + u_t X_t \sigma dW_t$$

**Objective:**

$$\max_u \quad E^P \left[ \Phi(X_T) \right]$$

**Information structure:**

- Complete information: We observe $S$ and $Y$, so $u \in \mathbf{F}$

- Incomplete information: We only observe $S$, so $u \in \mathbf{F}^S$. We need **filtering theory**.

# Filtering Theory – Setup

Given some filtration $\underline{\mathcal{F}}$:

$$\begin{aligned} dY_t &= a_t dt + dM_t \\ dZ_T &= b_t dt + dW_t \end{aligned}$$

Here all processes are $\mathbf{F}$ adapted and

$$\begin{aligned} Y &= \text{signal process,} \\ Z &= \text{observation process,} \\ M &= \text{martingale w.r.t. } \mathbf{F} \\ W &= \text{Wiener w.r.t. } \mathbf{F} \end{aligned}$$

We assume (for the moment) that $M$ and $W$ are **independent**.

**Problem:**
Compute (recursively) the filter estimate

$$\hat{Y}_t = \Pi_t[Y] = E\left[ Y_t \middle| \mathcal{F}_t^Z \right]$$

# The innovations process

Recall:

$$dZ_T = b_t dt + dW_t$$

Our best guess of $b_t$ is $\hat{b}_t$, so the genuinely new information should be

$$dZ_t - \hat{b}_t dt$$

The **innovations process** $\nu$ is defined by

$$\nu_t = dZ_t - \hat{b}_t dt$$

**Theorem:** The process $\nu$ is $\mathbf{F}^Z$-Wiener.

**Proof**: By Levy it is enough to show that

- $\nu$ is an $\mathbf{F}^Z$ martingale.
- $\nu_t^2 - t$ is an $\mathbf{F}^Z$ martingale.

**I. $\nu$ is an $\mathbf{F}^Z$ martingale:**

From definition we have

$$d\nu_t = \left(b_t - \hat{b}_t\right) dt + dW_t \tag{3}$$

so

$$E_s^Z\left[\nu_t - \nu_s\right] = \int_s^t E_s^Z\left[b_u - \hat{b}_u\right] du + E_s^Z\left[W_t - W_s\right]$$

$$= \int_s^t E_s^Z\left[E_u^Z\left[b_u - \hat{b}_u\right]\right] du + E_s^Z\left[E_s\left[W_t - W_s\right]\right] = 0$$

**I. $\nu_t^2 - t$ is an $\mathbf{F}^Z$ martingale:**

From Itô we have

$$d\nu_t^2 = 2\nu_t d\nu_t + \left(d\nu_t\right)^2$$

Here $d\nu$ is a martingale increment and from (3) it follows that $\left(d\nu_t\right)^2 = dt$.

**Remark 1:**
The innovations process gives us a Gram-Schmidt orthogonalization of the increasing family of Hilbert spaces

$$L^2(\mathcal{F}_t^Z); \quad t \geq 0.$$

**Remark 2:**
The use of Itô above requires general semimartingale integration theory, since we do not know a priori that $\nu$ is Wiener.

# Filter dynamics

From the $Y$ dynamics we guess that

$$d\hat{Y}_t = \hat{a}_t dt + \text{martingale}$$

**Definition:** $dm_t = d\hat{Y}_t - \hat{a}_t dt$.

**Proposition:** $m$ is an $\mathcal{F}_t^Z$ martingale.

**Proof:**

$$
\begin{aligned}
E_s^Z \left[ m_t - m_s \right] &= E_s^Z \left[ \hat{Y}_t - \hat{Y}_s \right] - E_s^Z \left[ \int_s^t \hat{a}_u du \right] \\
&= E_s^Z \left[ Y_t - Y_s \right] - E_s^Z \left[ \int_s^t \hat{a}_u du \right] \\
&= E_s^Z \left[ M_t - M_s \right] - E_s^Z \left[ \int_s^t \left( a_u - \hat{a}_u \right) du \right] \\
&= E_s^Z \left[ E_s \left[ M_t - M_s \right] \right] - E_s^Z \left[ \int_s^t E_u^Z \left[ a_u - \hat{a}_u \right] du \right] = 0
\end{aligned}
$$

# Filter dynamics

We now have the filter dynamics

$$d\hat{Y}_t = \hat{a}_t dt + dm_t$$

where $m$ is an $\mathcal{F}_t^Z$ martingale.

If the **innovations hypothesis**

$$\mathcal{F}_t^Z = \mathcal{F}_t^\nu$$

is true, then the martingale representation theorem would give us an $\mathcal{F}_t^Z$ adapted process $h$ such that

$$dm_t = h_t d\nu_t \tag{4}$$

The innovations hypothesis is not generally correct but FKK have proved that in fact (4) is always true.

# Filter dynamics

We thus have the filter dynamics

$$d\widehat{Y}_t = \widehat{a}_t dt + h_t d\nu_t$$

and it remains to determine the gain process $h$.

**Proposition:** The process $h$ is given by

$$h_t = \widehat{Y_t b_t} - \widehat{Y}_t \widehat{b}_t$$

We give a slighty heuristic proof.

# Proof scetch

From Itô we have

$$d\left(Y_t Z_t\right) = Y_t b_t dt + Y_t dW_t + Z_t a_t dt + Z_t dM_t$$

using

$$d\widehat{Y}_t = \widehat{a}_t dt + h_t d\nu_t$$

and

$$dZ_t = \hat{b}_t dt + d\nu_t$$

we have

$$d\left(\widehat{Y}_t Z_t\right) = \widehat{Y}_t \widehat{b}_t dt + \widehat{Y}_t d\nu_t + Z_t \widehat{a}_t dt + Z_t h_t d\nu_t + h_t dt$$

Formally we also should have

$$E\left[ d\left(Y_t Z_t\right) - d\left(\widehat{Y}_t X_t\right) \middle| \mathcal{F}_t^Z \right] = 0$$

which gives us

$$\left(\widehat{Y_t b_t} - \widehat{Y}_t \widehat{b}_t - h_t\right) dt = 0.$$

# The filter equations

For the model

$$
\begin{aligned}
dY_t &= a_t dt + dM_t \\
dZ_T &= b_t dt + dW_t
\end{aligned}
$$

where $M$ and $W$ are independent, we have the FKK non-linear filter equations

$$
\begin{aligned}
d\widehat{Y}_t &= \widehat{a}_t dt + \left\{ \widehat{Y_t b_t} - \widehat{Y}_t \widehat{b}_t \right\} d\nu_t \\
d\nu_t &= dZ_t - \hat{b}_t dt
\end{aligned}
$$

**Remark:** It is easy to see that

$$
h_t = E\left[ \left( Y_t - \widehat{Y}_t \right) \left( b_t - \widehat{b}_t \right) \middle| \mathcal{F}_t^Z \right]
$$

# The general filter equations

For the model

$$\begin{aligned} dY_t &= a_t dt + dM_t \\ dZ_T &= b_t dt + \sigma_t dW_t \end{aligned}$$

where

- The process $\sigma$ is $\mathcal{F}_t^Z$ adapted and positive.
- There is no assumption of independence between $M$ and $W$.

we have the filter

$$\begin{aligned} d\widehat{Y}_t &= \widehat{a}_t dt + \left[ \widehat{D}_t + \frac{1}{\sigma_t} \left\{ \widehat{Y_t b_t} - \widehat{Y}_t \widehat{b}_t \right\} \right] d\nu_t \\ d\nu_t &= \frac{1}{\sigma_t} \left\{ dZ_t - \hat{b}_t dt \right\} \\ dD_t &= \frac{d\langle M, W \rangle_t}{dt} \end{aligned}$$

# Comment on $\langle M, W \rangle$

This requires semimartingale theory but there are two simple cases

- If $M$ is Wiener then

$$d\langle M, W \rangle_t = dM_t dW_t$$

  with usual multiplication rules.

- If $M$ is a pure jump process then

$$d\langle M, W \rangle_t = 0.$$

# Filtering a Markov process

Assume that $Y$ is Markov with generator $G$. We want to compute $\Pi_t\left[f(Y_t)\right]$, for some nice function $f$. Dynkin's formula gives us

$$df(Y_t) = (Gf)(Y_t)dt + dM_t$$

Assume that the observations are

$$dZ_t = b(Y_t)dt + dW_t$$

where $W$ is independent of $Y$.

The filter equations are now

$$\begin{aligned} d\Pi_t[f] &= \Pi_t[Gf]\,dt + \{\Pi_t[fb] - \Pi_t[f]\,\Pi_t[b]\}\,d\nu_t \\ d\nu_t &= dZ_t - \Pi_t[b]\,dt \end{aligned}$$

**Remark:** To obtain $d\Pi_t[f]$ we need $\Pi_t[fb]$ and $\Pi_t[b]$. This leads generically to an infinite dimensional system of filter equations.

# On the filter dimension

$$d\Pi_t[f] = \Pi_t[Gf]\,dt + \{\Pi_t[fb] - \Pi_t[f]\,\Pi_t[b]\}\,d\nu_t$$

- To obtain $d\Pi_t[f]$ we need $\Pi_t[fb]$ and $\Pi_t[b]$.
- Thus we apply the FKK equations to $Gf$ and $b$.
- This leads to new filter estimates to determine and generically to an **infinite dimensional** system of filter equations.
- The filter equations are really equations for the entire conditional distribution of $Y$.
- You can only expect the filter to be finite when the conditional distribution of $Y$ is finitely parameterized.
- There are only very few examples of finite dimensional filters.
- The most well known finite filters are the Wonham and the Kalman filters.

# The Wonham filter

Assume that $Y$ is a continuous time Markov chain on the state space $\{1, \ldots, n\}$ with (constant) generator matrix $H$. Define the indicator processes by

$$\delta_i(t) = I\left\{Y_t = i\right\}, \quad i = 1, \ldots, n.$$

Dynkin's formula gives us

$$d\delta_t^i = \sum_j H(j,i)\delta_j dt + dM_t^i, \quad i = 1, \ldots, n.$$

Observations are

$$dZ_t = b(Y_t)dt + dW_t.$$

Filter equations:

$$d\Pi_t\left[\delta_i\right] = \sum_j H(j,i)\Pi_t\left[\delta_j\right]dt + \left\{\Pi_t\left[\delta_i b\right] - \Pi_t\left[\delta_i\right]\Pi_t\left[b\right]\right\}d\nu_t$$

$$d\nu_t = dZ_t - \Pi_t\left[b\right]dt$$

We note that

$$b(Y_t) = \sum_i b(i)\delta_i(t)$$

so

$$\begin{aligned} \Pi_t\left[\delta_i b\right] &= b(i)\Pi_t\left[\delta_i\right], \\ \Pi_t\left[b\right] &= \sum_j b(j)\Pi_t\left[\delta_j\right] \end{aligned}$$

We finally have the Wonham filter

$$\begin{aligned} d\widehat{\delta_i} &= \sum_j H(j,i)\widehat{\delta_j}dt + \left\{ b(i)\widehat{\delta_i} - \widehat{\delta_i}\sum_j b(j)\widehat{\delta_j} \right\} d\nu_t, \\ d\nu_t &= dZ_t - \sum_j b(j)\widehat{\delta_j}dt \end{aligned}$$

# The Kalman filter

$$\begin{aligned} dY_t &= aY_t dt + c dV_t, \\ dZ_t &= Y_t dt + dW_t \end{aligned}$$

$W$ and $V$ are independent Wiener

FKK gives us

$$\begin{aligned} d\Pi_t[Y] &= a\Pi_t[Y]dt + \left\{\Pi_t\left[Y^2\right] - (\Pi_t[Y])^2\right\} d\nu_t \\ d\nu_t &= dZ_t - \Pi_t[Y]dt \end{aligned}$$

We need $\Pi_t\left[Y^2\right]$, so use Itô to get write

$$dY_t^2 = \left\{2aY_t^2 + c^2\right\}dt + 2cY_t dV_t$$

From FKK:

$$\begin{aligned} d\Pi_t\left[Y^2\right] &= \left\{2a\Pi_t\left[Y^2\right] + c^2\right\}dt \\ &+ \left\{\Pi_t\left[Y^3\right] - \Pi_t\left[Y^2\right]\Pi_t[Y]\right\}d\nu_t \end{aligned}$$

Now we need $\Pi_t\left[Y^3\right]$! Etc!

Define the conditional error variance by

$$H_t = \Pi_t\left[\left(\Pi_t\left[Y_t\right] - \Pi_t\left[Y\right]\right)^2\right] = \Pi_t\left[Y^2\right] - \left(\Pi_t\left[Y\right]\right)^2$$

Itô gives us

$$d\left(\Pi_t\left[Y\right]\right)^2 = \left[2a\left(\Pi_t\left[Y\right]\right)^2 + H^2\right]dt + 2\Pi_t\left[Y\right]H d\nu_t$$

and Itô again

$$\begin{aligned} dH_t &= \left\{2aH_t + c^2 - H_t^2\right\}dt \\ &+ \left\{\Pi_t\left[Y^3\right] - 3\Pi_t\left[Y^2\right]\Pi_t\left[Y\right] + 2\left(\Pi_t\left[Y\right]\right)^3\right\}d\nu_t \end{aligned}$$

In **this particular case** we know (why?) that the distribution of $Y$ conditional on $Z$ is Gaussian!

Thus we have

$$\Pi_t\left[Y^3\right] = 3\Pi_t\left[Y^2\right]\Pi_t\left[Y\right] - 2\left(\Pi_t\left[Y\right]\right)^3$$

so $H$ is deterministic (as expected).

# The Kalman filter

Model:

$$
\begin{aligned}
dY_t &= aY_t dt + cdV_t, \\
dZ_t &= Y_t dt + dW_t
\end{aligned}
$$

Filter:

$$
\begin{aligned}
d\Pi_t\left[Y\right] &= a\Pi_t\left[Y\right]dt + H_t d\nu_t \\
\dot{H}_t &= 2aH_t + c^2 - H_t^2 \\
d\nu_t &= dZ_t - \Pi_t\left[Y\right]dt
\end{aligned}
$$

$$
H_t = \Pi_t\left[\left(\Pi_t\left[Y_t\right] - \Pi_t\left[Y\right]\right)^2\right]
$$

**Remark:** Because of the Gaussian structure, the conditional distribution evolves on a two dimensional submanifold. Hence a two dimensional filter.

# Optimal investment with stochastic rate of return

- A market model with a stochastic rate of return.
- Optimal portolios under complete information.
- Optimal portfolios under partial information.

# An investment problem with stochastic rate of return

**Model:**

$$dS_t = S_t\alpha(Y_t)dt + S_t\sigma dW_t$$

$W$ is scalar and $Y$ is some factor process. We assume that $(S, Y)$ is Markov and adapted to the filtration $\mathbf{F}$.

Wealth dynamics

$$dX_t = X_t \left\{ r + u_t \left[ \alpha(Y_t) - r \right] \right\} dt + u_t X_t \sigma dW_t$$

**Objective:**

$$\max_u \quad E^P \left[ X_T^\gamma \right]$$

We assume that $Y$ is a Markov process. with generator $\mathcal{A}$. We will treat several cases.

# A. Full information, $Y$ and $W$ independent.

The HJB equation for $F(t,x,y)$ becomes

$$F_t + \sup_u \left\{ u\left[\alpha - r\right] xF_x + rxF_x + \frac{1}{2}u^2x^2\sigma^2F_{xx} \right\} + \mathcal{A}F = 0$$

where $G$ operates on the $y$-variable. Obvious boundary condition

$$F(t,x,y) = x^\gamma$$

First order condition gives us:

$$\widehat{u} = \frac{r-\alpha}{x\sigma^2} \cdot \frac{F_x}{F_{xx}}$$

Plug into HJB:

$$F_t - \frac{(\alpha - r)^2}{2\sigma^2} \frac{F_x^2}{F_{xx}} + rxF_x + \mathcal{A}F = 0$$

**Ansatz:**

$$F(T,x,y) = x^{\gamma}G(t,y)$$

$$F_t = x^{\gamma}G_t, \quad F_x = \gamma x^{\gamma-1}G, \quad F_{xx} = \gamma(\gamma-1)x^{\gamma-2}G$$

Plug into HJB:

$$x^{\gamma}G_t + x^{\gamma}\frac{(\alpha-r)^2}{2\sigma^2}\beta G + r\gamma x^{\gamma}G + x^{\gamma}\mathcal{A}G = 0$$

where $\beta = \gamma/(\gamma-1)$.

$$\begin{aligned} G_t(t,y) + H(y)G(t,y) + \mathcal{A}G(t,y) &= 0, \\ G(T,y) &= 1. \end{aligned}$$

here

$$H(y) = r\gamma - \frac{[\alpha(y)-r]^2}{2\sigma^2}\beta$$

Kolmogorov gives us

$$H(y) = E_{t,y}\left[e^{\int_t^T H(Y_s)ds}\right]$$

# B. Full information, $Y$ and $W$ dependent.

Now we allow for dependence but restrict $Y$ to dynamics of the form.

$$\begin{aligned} dS_t &= S_t\alpha(Y_t)dt + S_t\sigma dW_t \\ dX_t &= X_t\left\{r + u_t\left[\alpha(Y_t) - r\right]\right\}dt + u_tX_t\sigma dW_t \\ dY_t &= a(Y_t)dt + b(Y_t)dW_t \end{aligned}$$

with **the same** Wiener process W driving both $S$ and $Y$. The imperfectly correlated case is a bit more messy but can also be handled.

HJB Equation for $F(t,x,y)$:

$$\begin{aligned} &F_t + aF_y + \frac{1}{2}b^2F_{yy} + rxF_x \\ &+ \sup_u\left\{u\left[\alpha - r\right]xF_x + \frac{1}{2}u^2x^2\sigma^2F_{xx} + uxb\sigma F_{xy}\right\} = 0. \end{aligned}$$

# Ansatz.

HJB:

$$F_t + aF_y + \frac{1}{2}b^2 F_{yy} + rxF_x$$
$$+ \sup_u \left\{ u\left[\alpha - r\right] xF_x + \frac{1}{2}u^2x^2\sigma^2 F_{xx} + uxb\sigma F_{xy} \right\} = 0.$$

After a lot of thinking we make the Ansatz

$$F(t,x,y) = x^\gamma h^{1-\gamma}(t,y)$$

and then it is not hard to see that $h$ satisfies a standard parabolic PDE. (Plug Ansatz into HJB).

See Zariphopoulou for details and more complicated cases.

# C. Partial information.

Model:

$$dS_t = S_t \alpha(Y_t) dt + S_t \sigma dW_t$$

**Assumption:** $Y$ cannot be observed directly.

**Reruirement:** The control $u$ must be $\mathcal{F}_t^S$ adapted. We thus have a partially observed system.

**Idea:** Project the $S$ dynamics onto the smaller $\mathcal{F}_t^S$ filtration and add filter equations in order to reduce the problem to the completely observable case.

Set $Z = \ln S$ and note (why?) that $\mathcal{F}_t^Z = \mathcal{F}_t^S$. We have

$$dZ_t = \left\{ \alpha(Y_t) - \frac{1}{2}\sigma^2 \right\} dt + \sigma dW_t$$

# Projecting onto the $S$-filtration

$$dZ_t = \left\{ \alpha(Y_t) - \frac{1}{2}\sigma^2 \right\} dt + \sigma dW_t$$

From filtering theory we know that

$$dZ_t = \left\{ \Pi_t \left[ \alpha \right] - \frac{1}{2}\sigma^2 \right\} dt + \sigma d\nu_t$$

where $\nu$ is $\mathcal{F}_t^S$-Wiener and

$$\Pi_t \left[ \alpha \right] = E \left[ \alpha(Y_t) \middle| \mathcal{F}_t^S \right]$$

We thus have the following $S$ dynamics on the $S$ filtration

$$dS_t = S_t \Pi_t \left[ \alpha \right] dt + S_t \sigma d\nu_t$$

and wealth dynamics

$$dX_t = X_t \left\{ r + u_t \left( \Pi_t \left[ \alpha \right] - r \right) \right\} dt + u_t X_t \sigma d\nu_t$$

# Reformulated problem

We now have the problem

$$\max_{u} E\left[X_T^{\gamma}\right]$$

for $Z$-adapted controls given wealth dynamics

$$dX_t = X_t \left\{ r + u_t \left( \Pi_t \left[ \alpha \right] - r \right) \right\} dt + u_t X_t \sigma d\nu_t$$

If we now can model $Y$ such that the (linear!) observation dynamics for $Z$ will produce a finite filter vector $\pi$, then **we are back in the completely observable case** with $Y$ replaced by $\pi$.and observation equation

We neeed a finite dimensional filter!

Two choices for $Y$

- Linear $Y$ dynamics. This will give us the Kalman filter. See Brendle
- $Y$ as a Markov chain. This will give us the Wonham filter. See Bäuerle and Rieder.

# Kalman case (Brendle)

Assume that

$$dS_t = Y_t S_t dt + S_t \sigma dW_t$$

with $Y$ dynamics

$$dY_t = aY_t dt + c dV_t$$

where $W$ and $V$ are independent. Observations:

$$dZ_t = \left\{ Y_t - \frac{1}{2}\sigma^2 \right\} dt + \sigma dW_t$$

We have a standard Kalman filter.

Wealth dynamics

$$dX_t = X_t \left\{ r + u_t \left( \widehat{Y}_t - r \right) \right\} dt + u_t X_t \sigma d\nu_t$$

# Kalman case, solution.

$$\max_u E\left[X_T^\gamma\right]$$

$$\begin{aligned} dX_t &= X_t\left\{r + u_t\left(\widehat{Y}_t - r\right)\right\}dt + u_t X_t \sigma d\nu_t \\ d\widehat{Y}_t &= \left\{\widehat{Y}_t - \frac{1}{2}\sigma^2\right\}dt + H_t \sigma d\nu_t \end{aligned}$$

where $H$ is deterministic and given by a Riccatti equation.

We are back in standard completly observable case with state variables $X$ and $\widehat{Y}$.

Thus the optimal value function is of the form

$$F(t, x, \hat{y}) = x^\gamma h^{1-\gamma}(t, \hat{y})$$

where $h$ solves a parabolic PDE and can be computed explicitly.