SF3940– PROBABILITY THEORY

SPRING 2016

HOMEWORK 1

DUE FEBRUARY 22, 2016

You must be able to explain the following concepts:

- The definition of a field (algebra), a σ-field (σ-algebra), a measure, properties of a measure, the extension theorem (Caratheodory's extension theorem), Dynkin's π-λ theorem (or the monotone class theorem), the Borel σ-field, measurable functions.

Solve at least *five* of the following problems *and* two problems of your own choice (either the remaining two below or two problems form the book you are using).

PROBLEM 1. Let $\mathscr{F}_n$ be classes of subsets of S. Suppose each $\mathscr{F}_n$ is a field, and $\mathscr{F}_n \subset \mathscr{F}_{n+1}$ for $n = 1, 2, \dots$. Define $\mathscr{F} = \cup_{n=1}^{\infty} \mathscr{F}_n$. Show that $\mathscr{F}$ is a field. Give an example to show that $\mathscr{F}$ need not be a σ-field.

PROBLEM 2. A (nonempty) collection $\mathscr{S}$ of subsets of Ω is called a semi-field (or semi-algebra) if it satisfies (i) $A, B \in \mathscr{S}$ implies $A \cap B \in \mathscr{S}$ and (ii) $A \in \mathscr{S}$ implies A^c is a finite disjoint union of sets in $\mathscr{S}$.

(a) Show that the collection of finite disjoint unions of sets in $\mathscr{S}$ is a field.
(b) Let $\Omega = (0, 1]$ and show that the collection of sets of the form $(a, b]$, $0 \le a < b \le 1$ and the empty set is a semi-field.

PROBLEM 3. Show that if $B \in \sigma(\mathscr{A})$, then there exists a countable subclass $\mathscr{A}_B$ of $\mathscr{A}$ such that $B \in \sigma(\mathscr{A}_B)$.

PROBLEM 4. Give an example of a measure μ on a σ-field $\mathscr{F}$ where there exists a monotone decreasing sequence $A_n \downarrow A \neq \varnothing$ such that $\mu(A_n) = \infty$ and $\mu(A) = 0$.

PROBLEM 5. Let μ^* be an outer measure on a sample space Ω. Show that if μ^* is finitely additive, then it is a measure.

PROBLEM 6. Show that the Borel σ-field on $\mathbb{R}$ is the smallest σ-field that makes all continuous functions $f : \mathbb{R} \to \mathbb{R}$ measurable. More precisely, let $\mathscr{R}$ denote the Borel σ-field and let $\mathscr{F}$ denote the smallest σ-field on $\mathbb{R}$ that makes all continuous functions $\mathscr{F}/\mathscr{R}$-measurable. Show that $\mathscr{F} = \mathscr{R}$.

PROBLEM 7. A function $f : \mathbb{R} \to \mathbb{R}$ is *lower semicontinuous* (l.s.c.) if $\liminf_{y \to x} f(y) \ge f(x)$ for all x. A function $f : \mathbb{R} \to \mathbb{R}$ is *upper semicontinuous* (u.s.c.) if $\limsup_{y \to x} f(y) \le f(x)$ for all x. Show that, if f is l.s.c. or u.s.c., then f is (Borel) measurable.

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