SF3940– PROBABILITY THEORY

SPRING 2016

HOMEWORK 4

DUE MARCH 21, 2016

You must be able to explain the following concepts:

- Weak convergence, the Portmanteau theorem, the continuous mapping theorem, Helly's selection theorem, characteristic functions and the inversion formula, the central limit theorem (Lindeberg-Feller thm and Berry-Essen), Poisson convergence, Cramér-Wold device.

Solve at least *four* of the following problems.

PROBLEM 1. Prove the following claim. If $X_1, X_2, \dots$ are independent and have characteristic function $e^{-|t|^\alpha}$, $0 < \alpha < 2$, then $n^{-1/\alpha}(X_1 + \cdots + X_n)$ has the same distribution as X_1. (This is an example of stable distributions).

PROBLEM 2. One can interpret the central limit theorem as the solution to a fixed point problem. Let $\mathcal{P}$ denote the set of probability measures μ on $\mathcal{R}$ with the properties that

$$\int x^2 \mu(dx) = 1, \quad \int x\mu(dx) = 0.$$

Define $T\mu$ for $\mu \in \mathcal{P}$ to be the probability measure given by

$$T\mu(B) = \int \int I\Big\{\frac{x+y}{\sqrt{2}} \in A\Big\}\mu(dx)\mu(dy), \quad \text{for Borel sets } B.$$

(a) Check that T maps $\mathcal{P}$ to itself.

(b) Use the central limit theorem to show that, for every $\mu \in \mathcal{P}$,

$$\lim_n \int \varphi dT^n\mu = \int \varphi d\gamma,$$

for each bounded continuous function φ, where γ is the standard normal distribution. Conclude that γ is the unique element of $\mathcal{P}$ with the property that $T\mu = \mu$ and that this fixed point is attracting.

PROBLEM 3. Let $X_n = (X_{n1}, X_{n2}, \dots, X_{nn})$ be uniformly distributed over the surface of a sphere with radius $\sqrt{n}$ in $\mathbb{R}^n$. Show that X_{n1} converges in distribution to a standard normal distribution.

PROBLEM 4. Consider PROBLEM 4 in Homework 2 where k balls are placed at random in n boxes. Suppose that $k = k(n)$ is such that $ne^{-k/n} \to \lambda$ as $n \to \infty$. Show that the distribution of the number of empty boxes converges to a Poisson distribution with parameter λ.

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PROBLEM 5. Let $X_1, X_2, \dots$ be iid with $EX_k = 0$ and $EX_k^2 = \sigma^2$. Show that

$$\frac{\sum_{k=1}^n X_k}{\left(\sum_{k=1}^n X_k^2\right)^{1/2}} \to \gamma,$$

in distribution, where γ is the standard normal distribution.

PROBLEM 6. Suppose $X_1, X_2, \dots$ are random variables such that the central limit theorem:

$$\sqrt{n}\frac{\bar{X}_n - c}{\sigma} \to \gamma,$$

in distribution, where γ is a standard normal distribution. Suppose f is a real smooth function with a nonzero derivative $f'(c)$ at c. Show that

$$\sqrt{n}\frac{f(\bar{X}_n) - f(c)}{\sigma|f'(c)|} \to \gamma,$$

(this is called the delta-method)

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