SF3940– PROBABILITY THEORY

SPRING 2016

HOMEWORK 6

DUE APRIL 18, 2016

You must be able to explain the following concepts:

- The Hahn decomposition, the Radon-Nikodym theorem, conditional probability, conditional expectation, properties of the conditional expectation

Solve *three* of the following problems **and** two additional problems (which may also be from the list). That is, five problems in total.

PROBLEM 1. Show by a counterexample that the Radon-Nikodym theorem ($\nu \ll \mu \Rightarrow d\nu/d\mu$ exists) does not hold if μ is not σ-finite.

PROBLEM 2. Let μ be the restriction of planar Lebesgue measure λ_2 to the σ-field $\mathcal{F} = \{A \times \mathbb{R} : A \in \text{Borel}(\mathbb{R})\}$ of vertical strips. Define ν on $\mathcal{F}$ by $\nu(A \times \mathbb{R}) = \lambda_2(A \times (0,1))$. Show that $\nu \ll \mu$ but has no density. Why does this not contradict the Radon-Nikodym theorem?

PROBLEM 3. (Borel's paradox) Suppose that a random point on the sphere is specified by longitude Θ and latitude Φ, in such a way that $0 \leq \Theta < \pi$, and $-\pi < \Phi \leq \pi$ (this is slightly different from the custom to let $0 \leq \Theta < 2\pi$ and $-\pi/2 < \Phi \leq \pi/2$). It may seem (at first) that if the point is chosen uniformly on the sphere then Θ and Φ should both be uniform over their possible values.

(a) Show that, given Φ, the conditional distribution of Θ is uniform over $[0, \pi)$
(b) Show that, given Θ, the conditional distribution of Φ has density $\frac{1}{4}|\cos\phi|$ over $(-\pi, \pi]$ (so it is not uniformly distributed).

PROBLEM 4. Of three prisoners, call them 1,2, and 3, two have been chosen by lot for execution. Prisoner 3 says to the guard, "Which of 1 and 2 is to be executed? One of them will be, and you give me no information about myself in telling me which it is". The guard finds this reasonable and says, "Prisoner 1 is to be executed". And now 3 reasons, "I know that 1 is to be executed; the other will be either 2 or me, and so my chance of being executed is now only 1/2, instead of the 2/3 it was before". Apparenbtly, the guard has given him information.

If one looks for a σ-field, it must be the one describing the guards answer, and it becomes clear that the sample space is incompletely specified. Suppose that, if 1 and 2 are to be executed the guard's response is "1" with probability p and "2" with probability $1-p$; and, of course, if 3 is to be executed the guard names the other victim. Calculate the conditional probabilities.

PROBLEM 5. (a) Generalize Markov's inequality: $P\{|X| \geq x \mid \mathcal{F}\} \leq x^{-k} E[|X|^k \mid \mathcal{F}]$ a.s.

(b) Similarly generalize two of the following indequalities: Chebyshev's, Jensen's, and Hölder's.

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PROBLEM 6. (a) Show that if $\mathcal{F} \subset \mathcal{G}$ and $EX^2 < \infty$, then

$$E[(X - E[X \mid G])^2] \leq E[(X - E[X \mid \mathcal{F})^2].$$

(b) Define $\mathrm{Var}(X \mid \mathcal{F}) = E[(X - E[X \mid \mathcal{F})^2 \mid \mathcal{F}]$. Prove that

$$\mathrm{Var}(X) = E[\mathrm{Var}(X \mid \mathcal{F})] + \mathrm{Var}(E[X \mid \mathcal{F}]).$$

PROBLEM 7. Let L^2 be the Hilbert space of random variables X on $(\Omega, \mathcal{F}, P)$ with $EX^2 < \infty$. For a σ-field $\mathcal{G} \subset \mathcal{F}$, let $M_{\mathcal{G}}$ be the subspace of elements in L^2 that are measurable $\mathcal{G}$. Show that the operator $P_{\mathcal{G}}$, defined for $X \in L^2$, by $X \mapsto E[X \mid \mathcal{G}]$ is the perpendicular projection on $M_{\mathcal{G}}$.

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