SF3940– PROBABILITY THEORY

SPRING 2016

HOMEWORK 7

DUE APRIL 25, 2016

You must be able to explain the following concepts:

- Martingale, sub/super-martingale, the martingale convergence theorem, Doob's decomposition, Doob's inequality, stopping times.

Solve *three* of the following problems **and** two additional problems (which may also be from the list). That is, five problems in total.

PROBLEM 1. Let $Z_1, Z_2, \dots$ be iid with $E|Z_1| < \infty$. Let θ be a random variable with finite mean and put $Y_i = \theta + Z_i$. If Z_i is $N(0,1)$ then, in statistics terms, we have a sample from a normal population with unknown mean and variance 1. The distribution of θ is called the *prior distribution* and $P\{\theta \in \cdot \mid Y_1, \dots, Y_n\}$ is called the *posterior distribution*. Show that

$$E[\theta \mid Y_1, \dots, Y_n] \to \theta, \text{almost surely.}$$

PROBLEM 2. Let $\{X_n\}$ be adapted to $\{\mathcal{F}_n\}$ with $0 \le X_n \le 1$. For $p \in (0,1)$, let $X_0 = x_0$ and

$$P\{X_{n+1} = p + (1-p)X_n \mid \mathcal{F}_n\} = X_n,\ P\{X_{n+1} = (1-p)X_n \mid \mathcal{F}_n\} = 1 - X_n.$$

Show that $X_n \to X$ a.s., where $P\{X = 1\} = x_0$ and $P\{X = 0\} = 1 - x_0$.

PROBLEM 3. Suppose $\mathcal{F}_n \uparrow \mathcal{F}$ and $Y_n \to Y$ in L_1 (i.e. $E[|Y_n - Y|] \to 0$). Show that $E[Y_n \mid \mathcal{F}_n] \to E[Y \mid \mathcal{F}_\infty]$ in L_1.

PROBLEM 4. Let S_n be the total assets of an insurance company at the end of year n. Suppose that in year n the company receives premiums of c and pays claims of the amount Z_n, where Z_n are independent $N(\mu, \sigma^2)$, with $0 < \mu < c$. The company is ruined if its assets ever fall below 0. Show that

$$P\{\text{ruin}\} \le \exp\Big(-2(c-\mu)S_0/\sigma^2\Big).$$

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PROBLEM 5. Suppose $\{Y_n\}$ are independent with $E|Y_n| < \infty$ and $EY_n = 0$. Show that for each $k \geq 1$ $\{X_n^{(k)}\}$ given by

$$X_n^{(k)} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} Y_{i_1} \cdots Y_{i_k}$$

is a $\mathbb{F}^{(k)}$-martingale, where $\mathbb{F}^{(k)} = \{\mathcal{F}_n^{(k)}\}$ is given by $\mathcal{F}_n^{(k)} = \sigma(X_1^{(k)}, \ldots, X_n^{(k)})$. Note that when $k = 2$, then $2X_n^{(2)} = S_n^2 - \sum_{i=1}^n Y_i^2$.

PROBLEM 6. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = \{\mathcal{F}_n\}$ a filtration. Let $\{X_n\}$ and $\{Y_n\}$ be positive, integrable, and $\mathbb{F}$-adapted. Suppose

$$E[X_{n+1} \mid \mathcal{F}_n] \leq (1 + Y_n) X_n \qquad \forall n$$

and $\sum_n Y_n < \infty$ a.s. Show that $\lim_n X_n$ exists a.s.

PROBLEM 7. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathbb{F} = \{\mathcal{F}_n\}$ a filtration. For a stopping time τ, let $\mathcal{F}_\tau$ be the information up to time τ. Formally $\mathcal{F}_\tau$ is the σ-field generated by all sets of the form $A \cap [\tau = n]$, $A \in \mathcal{F}_n$. Let $\{X_n\}$ be a sequence of integrable random variables.

(a) Show that $\{X_n\}$ is a $\mathbb{F}$-martingale if and only if for every stopping time $\tau \leq n$, (15%)

$$E[X_n \mid \mathcal{F}_\tau] = X_\tau.$$

(b) Show that if $\{X_n\}$ is a $\mathbb{F}$-martingale and τ is a bounded stopping time, then $E[X_\tau] = E[X_1]$.

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