# SOLUTION CONCEPTS FOR NORMAL-FORM GAMES PART A

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- Given: a *finite normal-form game* $G = (N, S, \pi)$, where
  - $N = \{1, ...n\}$ is the set of players
  - $S = \times_{i \in N} S_i$ is the set of (pure-)strategy profiles
  - $\pi : S \to \mathbb{R}^n$ is the combined payoff function

- Its mixed-strategy extension $\tilde{G} = (N, X, \tilde{\pi})$

- Examples: the (pure-strategy) normal, quasi-reduced normal, or educed normal forms of an EF game Γ

- Note that "nature" is not a player in the normal form

# 1 Nash's (1950) two interpretations

1. The *rationalistic* (or *epistemic*) interpretation:

   (a) All players are *rational* in the sense of Savage (Foundations of Statistics, 1954): players only use strategies that are optimal under *some* probabilistic belief about the other's strategy choices

   (b) If they randomize: statistical independence between different players' randomizations

   (c) All players *know* $G$. We may even assume that $G$ and all players' rationality is *common knowledge* (CK)

2. The *mass-action* (or *population-statistical*) interpretation:

"It is unnecessary to assume that the participants in a game have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal.

To be more detailed, we assume that there is a population (in the sense of statistics) of participants for each position of the game. Let us also assume that the 'average playing' of the game involves $n$ participants selected at random from the $n$ populations, and that there is a stable average frequency with which each pure strategy is employed by the 'average member' of the appropriate population.

Since there is to be no collaboration between individuals playing in different positions of the game, the probability that a particular $n$-tuple of pure strategies will be employed in playing of the game should be the product of the probabilities indicating the chance of each of the $n$ pure strategies to be employed in a random playing.

... Thus the assumptions we made in this 'mass action' interpretation led to the conclusion that the mixed strategies representing the average behavior in each of the populations form an equilibrium point." (*John Nash's (1950) PhD. thesis*)

- Today and next lecture: Focus on finite normal-form games!

- Let $G = (N, S, \pi)$ be a finite game with mixed-strategy extension $\tilde{G} = (N, \square(S), \tilde{\pi})$

- A *solution concept* is a correspondence that maps each game to a subset of its mixed-strategy polyhedron. A point in that subset is a *solution*.

# 2 The geometry of mixed-strategy spaces

- Let $S_i = \{1, ..., m_i\}$ be $i$'s pure strategies in $G$

- The player's *mixed-strategy simplex:*

$$X_i = \Delta_i = \Delta(S_i) = \{x_i \in \mathbb{R}_+^{m_i} : \sum_{h=1}^{m_i} x_{ih} = 1\}.$$

- The *vertices* of $\Delta_i$ are the *unit vectors,* $e_i^1, ..., e_i^{m_i} \in \mathbb{R}_+^{m_i}$

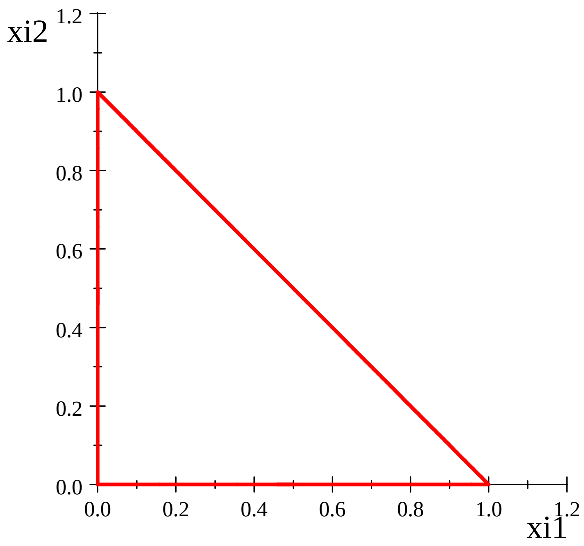
- The mixed-strategy simplex $\Delta_i$ is the *convex hull* of its vertices:

$$x_i = \sum_{h=1}^{m_i} x_{ih} e_i^h.$$

- Its (relative) interior: $int(\Delta_i) = \{x_i \in \Delta_i : x_{ih} > 0 \ \forall h \in S_i\}$

- Terminology: *interior* strategy $=$ *completely mixed* strategy

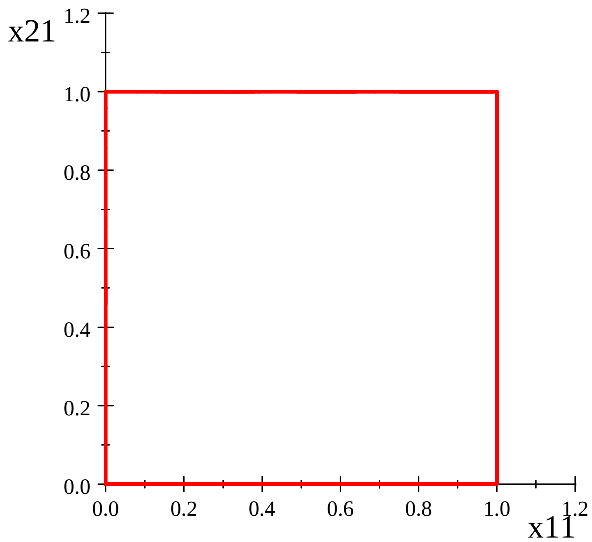
- Example: $|S_i| = 3$



- The *mixed-strategy polyhedron:*

$$X = \square = \square(S) = \times_{i \in N} \Delta_i = \times_{i \in N} \Delta(S_i)$$

- Example: $n = |S_1| = |S_2| = 2$



# 3 Mixed-strategy payoff functions

- They are all polynomial, actually *multi-linear*. For any players $i, j \in N$:

$$\tilde{\pi}_i(x) = \sum_{k=1}^{m_j} \tilde{\pi}_i(e_j^k, x_{-j}) \cdot x_{jk}$$

(the inner product between a constant vector in $\mathbb{R}^{m_i}$ and $x_j \in \mathbb{R}^{m_i}$)

- For $n = 2$ and $i = j = 1$:

$$\tilde{\pi}_1(x) = \sum_{h=1}^{m_1} \sum_{k=1}^{m_2} x_{1h} a_{hk} x_{2k} = x_1 \cdot A x_2$$

# 4 Best Replies and Dominance Relations

## 4.1 Best replies

- The $i$:th player's *pure-strategy best-reply correspondence* $\beta_i : \square(S) \rightrightarrows S_i$ is defined by

$$\beta_i(x) := \{h \in S_i : \tilde{\pi}_i(e_i^h, x_{-i}) \geq \tilde{\pi}_i(e_i^k, x_{-i}) \ \forall k \in S_i\}$$

- Mixed strategies cannot give higher payoffs than pure. (Why?) Hence

$$\beta_i(x) = \{h \in S_i : \tilde{\pi}_i(e_i^h, x_{-i}) \geq \tilde{\pi}_i(x_i', x_{-i}) \ \forall x_i' \in \Delta_i\}.$$

- The $i$:th player's *mixed-strategy best-reply correspondence* $\tilde{\beta}_i : \square(S) \rightrightarrows \Delta_i$ is defined by

$$\tilde{\beta}_i(x) := \{x_i^* \in \Delta_i : \tilde{\pi}_i(x_i^*, x_{-i}) \geq \tilde{\pi}_i(x_i', x_{-i}) \ \forall x_i' \in \Delta_i\}$$

- Note that

$$\tilde{\beta}_i(x) = \{x_i^* \in \boldsymbol{\Delta}_i : \textit{supp}(x_i^*) \subset \beta_i(x)\}$$

- $\tilde{\beta}_i(x)$ is a (non-empty) *face* of the simplex $\boldsymbol{\Delta}_i$. Hence: compact and convex!

- The *combined* **pure** *BR correspondence* $\beta : \square \rightrightarrows S$ is defined by

$$\beta(x) := \times_{i \in N} \beta_i(x)$$

- The *combined* **mixed** *BR correspondence* $\tilde{\beta} : \square \rightrightarrows \square$ is defined by

$$\tilde{\beta}(x) := \times_{i \in N} \tilde{\beta}_i(x)$$

## 4.2 Dominance relations

**Definition 4.1** $x_i^* \in \Delta_i$ **strictly dominates** $x_i' \in \Delta_i$ *if* $\tilde{\pi}_i(x_i^*, x_{-i}) > \tilde{\pi}_i(x_i', x_{-i})\ \forall x \in \Box$

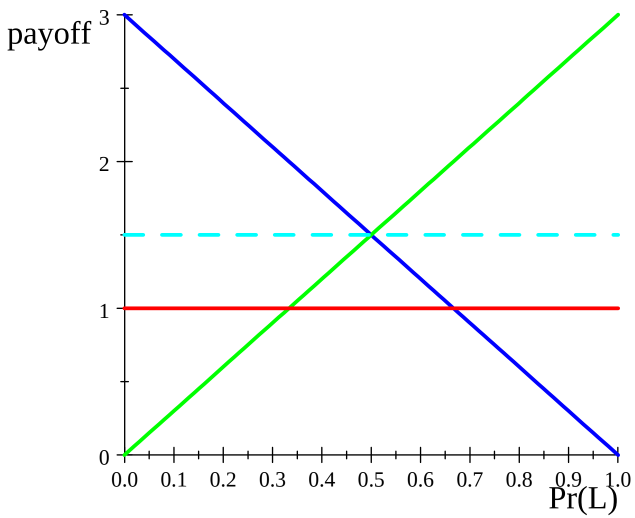
**Definition 4.2** $x_i^* \in \Delta_i$ **weakly dominates** $x_i' \in \Delta_i$ *if* $\tilde{\pi}_i(x_i^*, x_{-i}) \geq \tilde{\pi}_i(x_i', x_{-i})\ \forall x \in \Box$ *with* $>$ *for some* $x \in \Box$

**Definition 4.3** *A strategy that is not weakly dominated is* **undominated.**

- A strategy can be dominated without being dominated by any *pure* strategy

**Example 4.1** *Consider player 1 with payoff matrix*

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 1 & 1 \end{bmatrix}$$



**Proposition 4.1** *Each player has at least one undominated pure strategy.*

**Proof**:

1. Pick any player $i \in N$, $x^o \in int\,[\Box\,(S)]$ and $h \in \beta_i\,(x^o)$
2. Then $\tilde{\pi}_i(e_i^h, x_{-i}^o) \geq \tilde{\pi}_i(x_i, x_{-i}^o)\ \forall x_i \in \mathbf{\Delta}_i$
3. Claim: $\hat{x}_i = e_i^h$ is undominated
4. Suppose that $x_i^*$ weakly dominates $\hat{x}_i$
5. Then $\tilde{\pi}_i(x_i^*, s_{-i}) \geq \tilde{\pi}_i(\hat{x}_i, s_{-i})\ \forall s \in S$, with $>$ for some $s^o \in S$
6. $x^o$ attaches pos. proba. to $s^o \in S_j$, so $\tilde{\pi}_i(x_i^*, x_{-i}^o) > \tilde{\pi}_i(\hat{x}_i, x_{-i}^o)$. Contradiction.

- Iterated elimination of strictly dominated strategies:

  (a) Halts after a finite number of successive eliminations

  (b) The result is independent of the order of elimination

**Example 4.2** *A two-player game with payoff bi-matrix* $(A, B)$ *where* $B = A^T$ *and*

$$A = \begin{bmatrix} 3 & 1 & 6 \\ 0 & 0 & 4 \\ 1 & 2 & 5 \end{bmatrix}$$

- Property (b) not generally valid for iterated elimination of *weakly* dominated strategies

## 4.3 Dominance vs. best replies

- Pure best replies are not strictly dominated

**Q1:** If a pure strategy is *not* strictly dominated, is it then a best reply to *some* (mixed-)strategy profile?

- Pure best replies to *interior* strategy profiles are undominated

**Q2**: If a pure strategy is undominated, is it then a best reply to *some* interior belief?

**Proposition 4.2 (Pearce, 1984)** *Suppose* $n = 2$. *Then (a)* $h \in S_i$ *is not strictly dominated iff* $h \in \beta_i(x)$ *for some* $x \in \square$, *and (b)* $h \in S_i$ *is undominated iff* $h \in \beta_i(x)$ *for some* $x \in int(\square)$.

**Proof of claim (a):** [$h \in S_i$ strictly dominated] $\Leftrightarrow$ [$h \in \beta_i(x)$ for no $x \in \square$]

$\Rightarrow$: Trivial

$\Leftarrow$: Suppose that $h \in S_1$ not a BR to any $x$

1. Define $G^* = (N, \square(S), v)$ by

$$\begin{cases} v_1(x) := \tilde{\pi}_1(x_1, x_2) - \tilde{\pi}_1\left(e_1^h, x_2\right) \\ v_2(x) := -v_1(x) \end{cases}$$

2. Let $x^*$ be a NE of $G^*$ [$\exists$ by Nash's theorem]

3. Then $v_1(x^*) \geq v_1\left(e_1^h, x_2^*\right) = 0$, indeed $v_1(x^*) > 0$

4. $G^*$ zero-sum $\Rightarrow v_1\left(x_1^*, x_2\right) \geq v_1(x^*)\ \forall x_2 \in \Delta_2$

5. Thus $v_1\left(x_1^*, x_2\right) > 0\ \forall x_2 \in \Delta_2$. Equivalently:

$$\tilde{\pi}_1(x_1^*, x_2) > \tilde{\pi}_1\left(e_1^h, x_2\right) \quad \forall x_2 \in \Delta_2$$

Thus $h \in S_1$ is strictly dominated in the original game by $x_1^*$

- Not true for $n > 2$?

- Why?

# 5 Rationalizability

- Consider a finite game in normal form, $G = (N, S, \pi)$ and assume

  A1 (*Rationality)*: Each player $i$ forms a probabilistic belief $\mu_j^i \in \Delta\left(S_j\right)$ about every other player $j$'s strategy choice, a belief that does not contradict any information or knowledge that player $i$ has, and player $i$ chooses a (pure or mixed) strategy that maximize his or her expected payoff, assuming statistical independence between other player's strategy choices

  A2 (*Common knowledge*): The game $G$ and the players' rationality (A1) is *common knowledge* among the players

- We have observed that [A1 $\wedge$ A2] $\nRightarrow$ NE

**Q1**: What does A1 and A2 then imply?

**A1**: Rationalizability!

**Q2**: What is, then, “rationalizability”?

**A2**: The definition is a bit involved. We make it in steps

1. For any $X = \times_{j=1}^{n} X_j$, where each $X_j \subset \Delta\left(S_j\right)$, write

$$\tilde{\beta}_i(X) = \left\{x_i^* \in \Delta(S_i) : x_i^* \in \tilde{\beta}_i(x) \text{ for some } x \in X\right\}$$

2. Let $C^0 = \Box(S)$ and define $\left\langle C^t \right\rangle_{t \in \mathbb{N}}$ recursively by

$$\begin{cases} C_i^{t+1} = conv\left[\tilde{\beta}_i\left(C^t\right)\right] \ \forall i \in N \\ C^{t+1} = \times_{i \in N} C_i^{t+1} \end{cases}$$

3. Note that $C_i^{t+1} \subset C_i^t \ \forall t, i$

**Definition 5.1 (Pearce, 1984)** *A strategy* $x_i \in \Delta(S_i)$ *is* **rationalizable for player** $i$ *if* $x_i \in C_i^\infty$*, where*

$$C_i^\infty = \cap_{t \in \mathbb{N}} C_i^t.$$

**Proposition 5.1** $C_i^\infty = \Delta(R_i)$ *for some non-empty subset* $R_i \subset S_i$*.*

**Proof:**

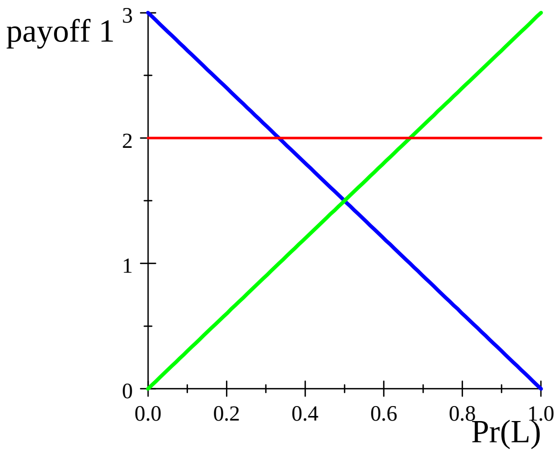
1. $\forall i, t$: $C_i^t$ is a non-empty subsimplex of $\Delta(S_i)$
2. The collection of subsimplices is finite

**Definition 5.2** *A pure strategy* $h \in S_i$ *is* **rationalizable** *for* $i$ *if* $h \in R_i$*.*

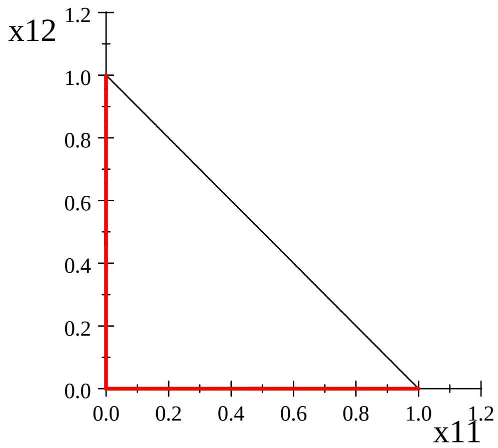
- The set $\tilde{\beta}_i\left(C^t\right) \subset \Delta\left(S_i\right)$ is not necessarily convex

**Example 5.1** *Consider player 1 with payoff matrix*

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \\ 2 & 2 \end{bmatrix}$$




*The subset* $\tilde{\beta}_1\left(C^0\right)$*:*




- Apply rationalizability to earlier examples!

# 6 The set of Nash equilibria

**Definition 6.1** $\square^{NE} = \left\{ x \in \square : x \in \tilde{\beta}(x) \right\}$

**Theorem 6.1 (Nash, 1950)** $\square^{NE} \neq \varnothing$*.*

- Two alternative proofs:

- Kakutani (Nash's first proof)

- Brouwer (Nash's second, and favorite, proof. Later used by Arrow and Debreu to prove $\exists$ of Walrasian equilibrium in their general equilibrium theory).

- The second proof (Nash, 1950b, 1951):

1. Let

$$u_{ih}(x) = \max\left\{0, \tilde{\pi}_i(e_i^h, x_{-i}) - \tilde{\pi}_i(x)\right\}$$

   The *excess payoff* of pure strategy $h \in S_i$ over strategy $x_i$

2. Increase your probability on pure strategies with positive excess payoff:

$$x_{ih}^{t+1} = f_{ih}\left(x^t\right) = \frac{x_{ih}^t + u_{ih}\left(x^t\right)}{1 + \sum_{k \in S_i} u_{ik}\left(x^t\right)} \qquad h \in S_i$$

3. This defines a continuous function $f : \square \to \square$. Each fixed point is a NE.

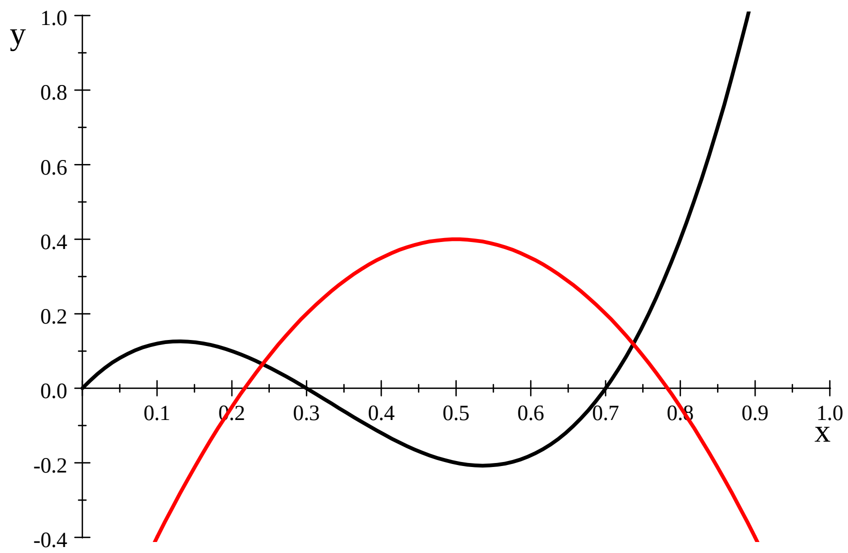
## 6.1 Topological structure

**Proposition 6.2** $\square^{NE}$ *is the finite union of disjoint, closed and connected sets.*

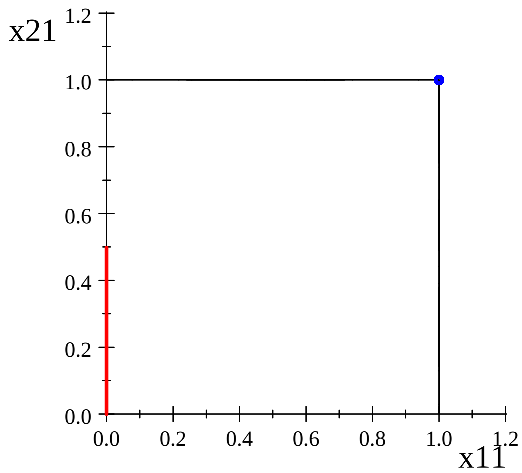
- These sets are called the Nash equilibrium *components* of the game $\tilde{G}$

**Proof of proposition:** Semialgebraic sets (van der Waerden, 1930s, Kohlberg and Mertens, 1986):

$$\square^{NE} = \left\{ x \in \square : \tilde{\pi}_i(x) - \tilde{\pi}_i(e_i^h, x_{-i}) \geq 0 \quad \forall i \in N, h \in S_i \right\}$$


y
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x

- Example: the Entry-Deterrence game



## 6.2 Invariance

1. Positive affine transformations of payoffs (this is well-known):

   Let $G = (N, S, \pi)$ and $G^* = (N, S, \pi^*)$

   where each $\pi_i^*$ is a positive affine transformation of $\pi_i$ $[\pi_i^*(s) \equiv a\pi_i(s) + b$, for $a > 0]$

2. Local payoff shifts (this is less known):

   Let $G = (N, S, \pi)$ and $G^* = (N, S, \pi^*)$

   where each $\pi_i^*$ is a *local payoff-shift* of $\pi_i$:

for some $\bar{s}_{-i} \in \times_{j \neq i} S_j$ and $c \in \mathbb{R}$, let

$$\pi_i^*(s) = \begin{cases} \pi_i(s) + c & \text{if } s_{-i} = \bar{s}_{-i} \\ \pi_i(s) & \text{otherwise} \end{cases}$$

**Proposition 6.3** *The set* $\square^{NE}$ *is unaffected by positive affine transformations and local payoff shifts.*

**Proof of second claim**: Note that

$$\pi_i^*(x_i', x_{-i}) - \pi_i^*(x_i'', x_{-i}) \equiv \tilde{\pi}_i(x_i', x_{-i}) - \tilde{\pi}_i(x_i'', x_{-i})$$

- These invariance properties are extremely helpful when looking for NE.

**Example 6.1**

$$\begin{array}{ccc} & L & R \\ T & 7,9 & 1,6 \\ B & 5,2 & 2,3 \end{array} \sim \begin{array}{ccc} & L & R \\ T & 2,9 & 0,6 \\ B & 0,2 & 1,3 \end{array} \sim \begin{array}{ccc} & L & R \\ T & 2,3 & 0,0 \\ B & 0,0 & 1,1 \end{array}$$

Next topic: (a) refining the Nash equilibrium concept, and (b) going set-wise

[Lecture notes chapter 4, chapter 1 in EGT book.]

# THE END