SOLUTION CONCEPTS FOR NORMAL-FORM GAMES
PART B

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February 19, 2010
1 Refining the Nash equilibrium concept

- While NE does not follow from CK[game+rationality],

- NE is necessary for rational players with shared expectations in the sense that

\[ K[\text{game+rationality}] \land [x^* \in \square (S) \text{ expected}] \Rightarrow x^* \in \square^{NE} \]

- However, this does not imply that \( x^* \) will be played, unless \( x^* \) is strict - you may expect others to play \( x^* \) and yet use another best reply yourself

- Given some time to “learn” to play a game (protocol), in many applications, human subjects tend towards NE play
• Moreover, as we will later see, evolutionary processes tend to lead to NE (or certain sets that contain NE)
• In addition to the question of whether we should expect NE play: are all NE “equally good” as predictions?

• Recall the entry-deterrence game

1. To use a *weakly* dominated strategy is like not taking an insurance that is available for free, an insurance against all eventualities associates with all other players’ strategy choices. At least in simultaneous-move games, it would seem unwise not to take such an insurance.

2. However, there are Nash equilibria with weakly dominated strategies

3. Recall the entry-deterrence game, and consider

\[
\begin{array}{c|cc}
& L & R \\
T & 1,1 & 0,0 \\
B & 0,0 & 0,0 \\
\end{array}
\]
4. Question 1: Does every finite game have a (pure or mixed) NE in undominated strategies?

5. An epistemic puzzle: If “prudent rationality” means to use only undominated best replies to one’s (probabilistic) beliefs, can then “prudent rationality” be CK? The condition that a player is “prudently rationality” seems to conflict with the condition that he/she knows that another player is “prudently rationality” [Samuelson (1992), Brandenburger, Friedenberg and Keisler (2008)]

6. Question 2: Can one formulate a consistent model of “bounded rationality” and what will it lead to? NE? A selection of NE? Nothing?
1.1 Perfection

• Focus on finite games, and their mixed-strategy extensions

• “Rationality as the limit of bounded rationality when the bounds vanish”

• Selten (1975): All players have “trembling hands” and know this

• Strategy perturbations:

\[ x_i \rightarrow \bar{x}_i = (1 - \varepsilon_i) x_i + \varepsilon_i q_i \]

for error probability \( \varepsilon_i \in (0, 1) \) and conditional mistake distribution \( q_i \in \text{int} (\Delta_i) \)
Figure 1:
The resulting distribution $\bar{x}_i$ is a mixed strategy, a “compound lottery” over $S_i$:

Note $\bar{x}_{ih} \geq \lambda_{ih}$ where $\lambda_{ih} = \varepsilon_i q_{ih} \ \forall i \in N, h \in S_i$

Equivalent, for player $i$, with choosing $\bar{x}_i$ directly, in subset

$$X_i (\lambda) = \{ \bar{x}_i \in \Delta (S_i) : \bar{x}_{ih} \geq \lambda_{ih} \ \forall h \in S_i \}$$

where

$$\lambda \in \Lambda = \left\{ (\lambda_{ih}) : \lambda_{ih} > 0 \ \forall i, h \text{ and } \sum_{h \in S_i} \lambda_{ih} < 1 \ \forall i \right\}$$

For any $\lambda \in \Lambda$: the associated perturbed game $\tilde{G} (\lambda) = (N, X(\lambda), \tilde{\pi})$

- an infinite (Euclidean) game, with compact, convex strategy sets
Figure 2:
Let \( X^{NE}(\lambda) \subset X(\lambda) \) be the set of Nash equilibria of \( \tilde{G}(\lambda) \).

**Proposition 1.1** \( X^{NE}(\lambda) \neq \emptyset \) for all \( \lambda \in \Lambda \).

**Proof:** For each \( i \in N \) and \( \lambda \in \Lambda \), define \( \tilde{\beta}_i^\lambda : X(\lambda) \Rightarrow X_i(\lambda) \) by

\[
\tilde{\beta}_i^\lambda(x) = \left\{ x_i^* \in X_i(\lambda) : \tilde{\pi}_i(x_i^*, x_{-i}) \geq \tilde{\pi}_i(x_i', x_{-i}) \ \forall x_i' \in X_i(\lambda) \right\}
\]

and apply Kakutani’s Fixed-Point Theorem.

**Definition 1.1 (Selten, 1975)** Let \( G = (N, S, \pi) \) be a finite game. \( x^* \in \Box^{NE} \) is perfect if, for some sequence of perturbed games \( \tilde{G}^{\lambda t} \) with \( (\lambda^t)_{t \in \mathbb{N}} \rightarrow 0 \), there exists an accompanying sequence \( x^t \) of Nash equilibria \( x^t \) in \( \tilde{G}^{\lambda t} \) such that \( x^t \rightarrow x^* \).

**Notation:** \( \Box^{PE} \subset \Box^{NE} \)
Theorem 1.2 (Selten, 1975) $\square^{PE} \neq \emptyset$.

Proof: In class (using the Bolzano-Weierstrass Theorem).

- Note the word “some” in the definition. Can it be strengthened to “all” without losing existence in some games?

Example 1.1

$$
\begin{array}{ccc}
L & M & R \\
T & 3,3 & 1,0 & 0,2 \\
B & 3,3 & 0,2 & 1,0 \\
\end{array}
$$

- Note that $\square^{NE} \cap int(\square) \subset \square^{PE}$
A characterization:

\[
\text{perfection } \iff \text{“robustness to strategic uncertainty”}
\]

Proposition 1.3 (Selten, 1975) \( x^* \in \Box^{PE} \) iff every neighborhood of \( x^* \) contains some \( x^o \in \text{int}(\Box) \) such that \( x^* \in \tilde{\beta}(x^o) \).

Corollary 1.4 If \( x \in \Box^{PE} \) then \( x \in \Box^{NE} \) is undominated, and, if \( n = 2 \), then the converse is true.

Proof:

1. \( x \in \Box^{PE} \Rightarrow x^* \in \tilde{\beta}(x^o) \) for some \( x^o \in \text{int}(\Box) \) \( \Rightarrow x^* \) undominated
2. For $n = 2$ and $x^* \in \square^{NE}$ undominated: $x^* \in \tilde{\beta}(x^o)$ for some $x^o \in \text{int}(\square)$. For all $\varepsilon \in (0, 1)$, let

$$x^\varepsilon = (1 - \varepsilon)x^* + \varepsilon x^o$$

Then $x^\varepsilon \in \text{int}(\square)$ and $x^* \in \tilde{\beta}(x^\varepsilon) \forall \varepsilon \in (0, 1)$ (by bilinearity)

Example 1.2 *The coordination game:*

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>2,2</td>
<td>0,0</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Example 1.3 *The entry-deterrence game:*

<table>
<thead>
<tr>
<th></th>
<th>Y</th>
<th>F</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1,3</td>
<td>1,3</td>
</tr>
<tr>
<td>E</td>
<td>2,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>
Roger Myerson discovered that *non-perfect* equilibria may *become perfect* if we add a strictly dominated strategy to the game - a perhaps not so desirable property of the solution concept?

**Example 1.4** Add a "dumb" strategy to the entry-deterrence game:

<table>
<thead>
<tr>
<th></th>
<th>( C )</th>
<th>( F )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>1, 3</td>
<td>1, 3</td>
</tr>
<tr>
<td>( E )</td>
<td>2, 2</td>
<td>0, 0</td>
</tr>
<tr>
<td>( D )</td>
<td>−1, −1</td>
<td>−1, 0</td>
</tr>
</tbody>
</table>

Now \((A, F)\) is perfect! Because if player 1 would play \( D \) by mistake, then \( F \) is better than \( C \).
1.2 Properness

- Myerson (1978): Players have “trembling hands,” know this, and try to avoid more costly mistakes more than less costly mistakes.

- Require robustness against trembles such that more “costly” mistakes are an order of magnitude less probable than less “costly” ones: \( \varepsilon, \varepsilon^2, \ldots, \varepsilon^k, \ldots \)

**Definition 1.2 (Myerson, 1978)** Given \( \varepsilon > 0 \), a strategy profile \( x \in \square(S) \) is \( \varepsilon \)-proper if \( x \in \text{int}[\square(S)] \) and

\[
\tilde{\pi}_i\left(e^h_i, x_{-i}\right) < \tilde{\pi}_i\left(e^k_i, x_{-i}\right) \Rightarrow x_{ih} \leq \varepsilon \cdot x_{ik},
\]

**Definition 1.3 (Myerson, 1978)** \( x^* \in \square^{NE} \) is proper if, for some sequence \( \varepsilon_t \rightarrow 0 \) there exist \( \varepsilon_t \)-proper profiles \( x^t \rightarrow x^* \).
• Let $\square^{PR} \subset \square^{NE}$ denote the set of proper equilibria.

• Note that any $x \in \square^{NE} \cap \text{int}(\square)$ is $\varepsilon$-proper for all $\varepsilon > 0$ and hence:

$\square^{NE} \cap \text{int}(\square) \subset \square^{PR}$

Proposition 1.5 (Myerson, 1978) $\square^{PR} \neq \emptyset$.

Proof idea:

1. Sufficient to show that there for every $\varepsilon \in (0, 1)$ exists some $\varepsilon$-proper strategy profile (and then use Bolzano-Weierstrass)

2. For $\varepsilon > 0$ fixed, define $\varphi^{\varepsilon} : \square(S) \Rightarrow \square(S)$ by $\varphi^{\varepsilon}(x) = \times_{i=1}^{n} \varphi_{i}^{\varepsilon}(x)$ where

$$\varphi_{i}^{\varepsilon}(x) = \left\{ x_{i}^{*} \in \Delta(S_{i}) : x_{ih}^{*} \leq \varepsilon x_{ik}^{*} \text{ if } \tilde{\pi}_{i}(e_{i}^{h}, x_{-i}) < \tilde{\pi}_{i}(e_{i}^{k}, x_{-i}) \right\}$$
3. Convex-valued, compact-valued and u.h.c. Apply Kakutani to $\varphi^\varepsilon$

4. Each fixed point under $\varphi^\varepsilon$ is $\varepsilon$-proper
Example 1.5  Reconsider the expanded entry-deterrence game, with a “dumb” strategy. We saw that \((A, F)\) is perfect. But is it proper? No, because mistake \(D\) is more costly than mistake \(E\).

Remark 1.1  van Damme showed that also the set of proper equilibria can change when one adds a strictly dominated strategy to a game. However, van Damme also showed that properness has a remarkable and beautiful implication for extensive-form analysis (later lecture!)
1.3 Essentiality

- Discard NE that are not robust to perturbations of payoffs!

- The players know the payoffs, but the analyst is not absolutely sure

- Given $G = (N, S, \pi), \pi : S \to \mathbb{R}^n$, consider all games $G^* = (N, S, \pi^*)$, for $\pi^* : S \to \mathbb{R}^n$

- Define distance between games $G$ and $G^*$:

$$d(G, G^*) = \max_{i \in N, s \in S} | \pi_i(s) - \pi_i^*(s) | .$$

**Definition 1.4 (Wu and Jiang, 1962)** $x \in \square^N$ is essential if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every game $G^*$ within payoff distance $\delta$ from $G$ has a Nash equilibrium within distance $\varepsilon$ from $x$. 
• Based on Fort (1950), “essential fixed points under continuous mappings”

**Proposition 1.6** *Essentiality* $\Rightarrow$ *Perfection*

**Proof idea:** Trembles can be viewed as special forms of payoff perturbations.
However:

- Does still not reject the mixed equilibrium in

\[
\begin{array}{ccc}
L & R \\
T & 2, 2 & 0, 0 \\
B & 0, 0 & 1, 1 \\
\end{array}
\]

- and some games have no essential equilibria at all:

\[
\begin{array}{ccc}
L & R \\
T & 1, 2 & 0, 0 \\
B & 1, 3 & 0, 0 \\
\end{array}
\]
2 Set-valued solutions

- The EF generic constancy of outcomes on NE components (Kreps and Wilson, 1982)

2.1 Essential NE components

Definition 2.1 (Jiang, 1963) A non-empty closed set $X \subset \square^{NE}$ is essential if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every game $G^*$ within payoff distance $\delta$ from $G$ has a Nash equilibrium within distance $\varepsilon$ from $X$.

Proposition 2.1 Let $G$ be a finite game. At least one component of the set $\square^{NE}$ is essential.
Proposition 2.2 Let $G$ be a finite game. Every essential component of $\square^{NE}$ contains at least one strategically stable set.

- However, we still do not get rid of the mixed NE in the coordination game

\[
\begin{array}{cc}
L & R \\
T & 2,2 & 0,0 \\
B & 0,0 & 1,1 \\
\end{array}
\]
2.2 Strategic stability

- Elon Kohlberg and Jean-Francois Mertens (1986) showed that any point-valued solution concept, no matter how defined, will have some theoretical flaw.

- They therefore suggested that we instead define solutions set-wise.

- Instead of considering a strategy profile $x \in \square(S)$ consider a (closed) subset $X \subset \square(S)$

**Definition 2.2 (Kohlberg-Mertens, 1986)** $X \subset \square(S)$ is strategically stable if it is minimal with respect to the following property: $X$ is non-empty and closed, and for every $\varepsilon > 0$ there is some $\delta > 0$ such that every strategy-perturbed game $G(\lambda) = (N, X(\lambda), u)$ with mistake probabilities $\lambda_{ih} < \delta$ has some Nash equilibrium within distance $\varepsilon$ from $X$. 
• Minimality means that the set has no proper subset with the stated property

• One can show that if $X$ is strategically stable, then $X \subset \square^{PE}$.

• However, we still do not get rid of the mixed NE in the coordination game

• And, also disturbing is that strategically stable sets may contain equilibria from different NE components

• For a fix of this and other theoretical flaws, see Mertens (1989)
• Next lecture: Solution concepts for finite EF games, and relations to NF solutions

THE END