SOLUTION CONCEPTS FOR NORMAL-FORM GAMES PART B

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1 Refining the Nash equilibrium concept

- While NE does not follow from CK[game+rationality],

- NE is *necessary* for rational players *with shared expectations* in the sense that

$$\text{K[game+rationality]} \wedge [x^* \in \square(S) \text{ expected}] \Rightarrow x^* \in \square^{NE}$$

- However, this does not imply that x^* *will be played*, unless x^* is strict - you may expect others to play x^* and yet use another best reply yourself

- Given some time to "learn" to play a game (protocol), in many applications, human subjects tend towards NE play

- Moreover, as we will later see, evolutionary processes tend to lead to NE (or certain sets that contain NE)

- In addition to the question of whether we should expect NE play: are all NE "equally good" as predictions?

- Recall the entry-deterrence game

1. To use a *weakly* dominated strategy is like not taking an insurance that is available for free, an insurance against all eventualities associates with all other players' strategy choices. At least in simultaneous-move games, it would seem unwise not to take such an insurance.

2. However, there are Nash equilibria with weakly dominated strategies

3. Recall the entry-deterrence game, and consider

	L	R
T	$1, 1$	$0, 0$
B	$0, 0$	$0, 0$

4. Question 1: Does every finite game have a (pure or mixed) NE in undominated strategies?

5. An epistemic puzzle: If “prudent rationality” means to use only undominated best replies to one’s (probabilistic) beliefs, can then “prudent rationality” be CK? The condition that a player is “prudently rationality” seems to conflict with the condition that he/she knows that another player is “prudently rationality” [Samuelson (1992), Brandenburger, Friedenberg and Keisler (2008)]

6. Question 2: Can one formulate a consistent model of “bounded rationality” and what will it lead to? NE? A selection of NE? Nothing?

1.1 Perfection

- Focus on finite games, and their mixed-strategy extensions

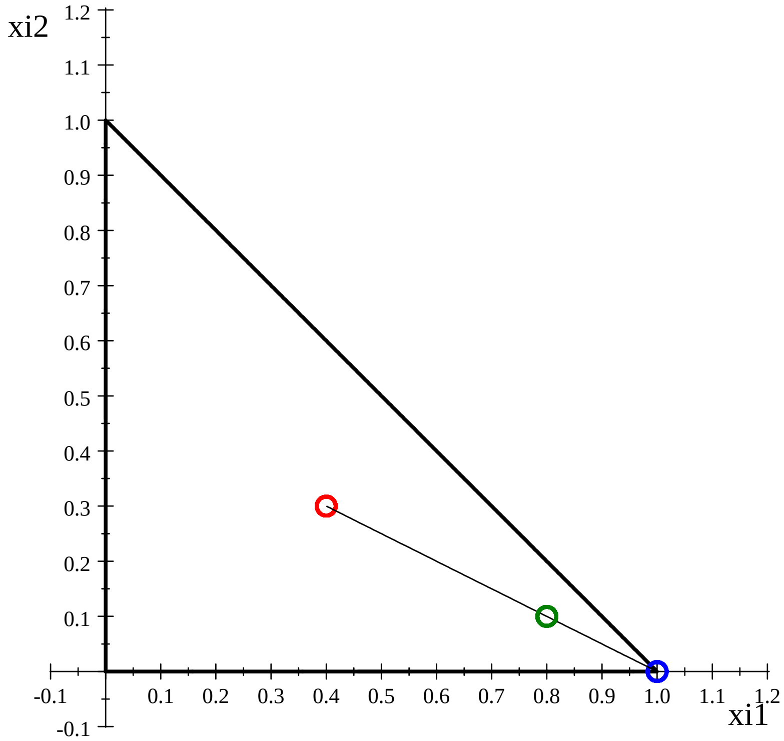
- "Rationality as the limit of bounded rationality when the bounds vanish"

- Selten (1975): All players have "trembling hands" and know this

- Strategy perturbations:

$$x_i \to \bar{x}_i = (1 - \varepsilon_i) x_i + \varepsilon_i q_i$$

  for *error probability* $\varepsilon_i \in (0, 1)$ and *conditional mistake distribution* $q_i \in int(\Delta_i)$



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xi1

Figure 1:

- The resulting distribution $\bar{x}_i$ is a mixed strategy, a "compound lottery" over S_i:

- Note $\bar{x}_{ih} \geq \lambda_{ih}$ where $\lambda_{ih} = \varepsilon_i q_{ih}\ \forall i \in N, h \in S_i$

- Equivalent, for player i, with choosing $\bar{x}_i$ directly, in subset

$$X_i(\lambda) = \{\bar{x}_i \in \Delta(S_i) : \bar{x}_{ih} \geq \lambda_{ih} \quad \forall h \in S_i\}$$

  where

$$\lambda \in \Lambda = \left\{(\lambda_{ih}) : \lambda_{ih} > 0\ \forall i, h \text{ and } \sum_{h \in S_i} \lambda_{ih} < 1\ \forall i\right\}$$

- For any $\lambda \in \Lambda$: the associated *perturbed game* $\tilde{G}(\lambda) = (N, X(\lambda), \tilde{\pi})$

  - an infinite (Euclidean) game, with compact, convex strategy sets

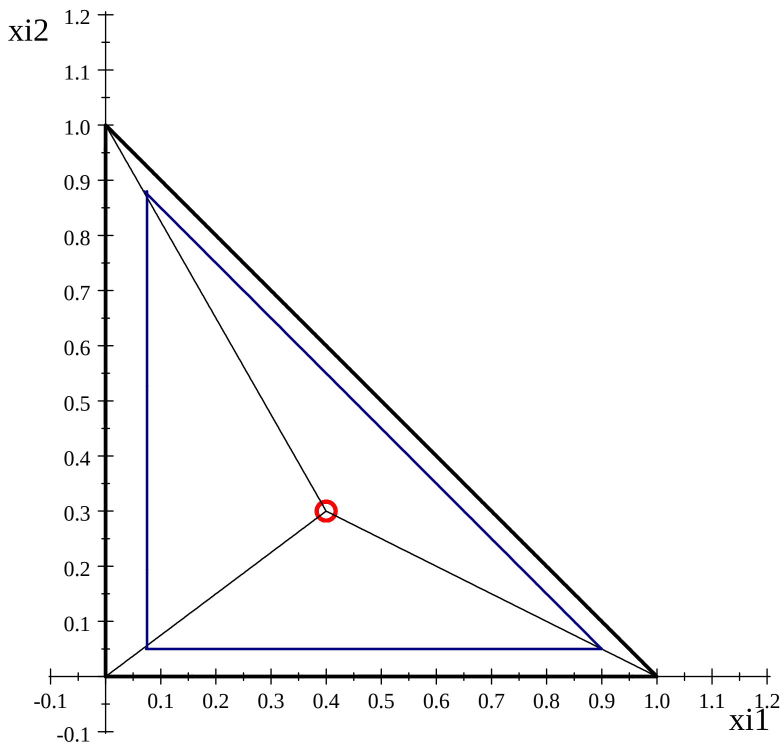

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Figure 2:

- Let $X^{NE}(\lambda) \subset X(\lambda)$ be the set of Nash equilibria of $\tilde{G}(\lambda)$

Proposition 1.1 $X^{NE}(\lambda) \neq \varnothing$ *for all* $\lambda \in \Lambda$.

Proof: For each $i \in N$ and $\lambda \in \Lambda$, define $\tilde{\beta}_i^{\lambda} : X(\lambda) \rightrightarrows X_i(\lambda)$ by

$$\tilde{\beta}_i^{\lambda}(x) = \left\{ x_i^* \in X_i(\lambda) : \tilde{\pi}_i(x_i^*, x_{-i}) \geq \tilde{\pi}_i(x_i', x_{-i})\ \forall x_i' \in X_i(\lambda) \right\}$$

and apply Kakutani's Fixed-Point Theorem.

Definition 1.1 (Selten, 1975) *Let* $G = (N, S, \pi)$ *be a finite game.* $x^* \in \square^{NE}$ *is* **perfect** *if, for some sequence of perturbed games* $\left\langle \tilde{G}^{\lambda^t} \right\rangle_{t \in \mathbb{N}}$ *with* $\left\langle \lambda^t \right\rangle_{t \in \mathbb{N}} \to 0$*, there exists an accompanying sequence* $\left\langle x^t \right\rangle_{t \in \mathbb{N}}$ *of Nash equilibria* x^t *in* $\tilde{G}^{\lambda^t}$ *such that* $x^t \to x^*$.

- Notation: $\square^{PE} \subset \square^{NE}$

Theorem 1.2 (Selten, 1975) $\square^{PE} \neq \varnothing$.

Proof: In class (using the Bolzano-Weierstrass Theorem).

- Note the word "some" in the definition. Can it be strengthened to "all" without losing existence in some games?

Example 1.1

	L	M	R
T	3, 3	1, 0	0, 2
B	3, 3	0, 2	1, 0

- Note that $\square^{NE} \cap int(\square) \subset \square^{PE}$

- A characterization:

  perfection $\Leftrightarrow$ "robustness to strategic uncertainty"

Proposition 1.3 (Selten, 1975) *$x^* \in \square^{PE}$ iff every neighborhood of x^* contains some $x^o \in int(\square)$ such that $x^* \in \tilde{\beta}(x^o)$.*

Corollary 1.4 *If $x \in \square^{PE}$ then $x \in \square^{NE}$ is undominated, and, if $n = 2$, then the converse is true.*

Proof:

1. $x \in \square^{PE} \Rightarrow x^* \in \tilde{\beta}(x^o)$ for some $x^o \in int(\square) \Rightarrow x^*$ undominated

2. For $n = 2$ and $x^* \in \square^{NE}$ undominated: $x^* \in \tilde{\beta}(x^o)$ for some $x^o \in int(\square)$. For all $\varepsilon \in (0,1)$, let

$$x^\varepsilon = (1-\varepsilon)\,x^* + \varepsilon x^o$$

Then $x^\varepsilon \in int(\square)$ and $x^* \in \tilde{\beta}(x^\varepsilon)\ \forall\varepsilon \in (0,1)$ (by bilinearity)

Example 1.2 *The coordination game:*

	L	R
T	$2,2$	$0,0$
B	$0,0$	$1,1$

Example 1.3 *The entry-deterrence game:*

	Y	F
A	$1,3$	$1,3$
E	$2,2$	$0,0$

- Roger Myerson discovered that *non-perfect* equilibria may *become perfect* if we add a strictly dominated strategy to the game - a perhaps not so desirable property of the solution concept?

Example 1.4 *Add a "dumb" strategy to the entry-deterrence game:*

	C	F
A	$1, 3$	$1, 3$
E	$2, 2$	$0, 0$
D	$-1, -1$	$-1, 0$

Now (A, F) *is perfect! Because if player 1 would play D by mistake, then F is better than C.*

1.2 Properness

- Myerson (1978): Players have "trembling hands," know this, and *try to avoid more costly mistakes* more than less costly mistakes

- Require robustness against trembles such that more "costly" mistakes are *an order of magnitude* less probable than less "costly" ones: ε, ε^2,...,ε^k,...

Definition 1.2 (Myerson, 1978) *Given $\varepsilon > 0$, a strategy profile $x \in \square(S)$ is ε-**proper** if $x \in int[\square(S)]$ and*

$$\tilde{\pi}_i\left(e_i^h, x_{-i}\right) < \tilde{\pi}_i\left(e_i^k, x_{-i}\right) \quad \Rightarrow \quad x_{ih} \leq \varepsilon \cdot x_{ik} \ ,$$

Definition 1.3 (Myerson, 1978) *$x^* \in \square^{NE}$ is **proper** if, for some sequence $\varepsilon_t \to 0$ there exist ε_t-proper profiles $x^t \to x^*$.*

- Let $\square^{PR} \subset \square^{NE}$ denote the set of proper equilibria.

- Note that any $x \in \square^{NE} \cap int(\square)$ is ε-proper for all $\varepsilon > 0$ and hence:

$$\square^{NE} \cap int(\square) \subset \square^{PR}$$

Proposition 1.5 (Myerson, 1978) $\square^{PR} \neq \varnothing$.

Proof idea:

1. Sufficient to show that there for every $\varepsilon \in (0,1)$ exists some ε-proper strategy profile (and then use Bolzano-Weierstrass)

2. For $\varepsilon > 0$ fixed, define $\varphi^{\varepsilon} : \square(S) \rightrightarrows \square(S)$ by $\varphi^{\varepsilon}(x) = \times_{i=1}^{n} \varphi_i^{\varepsilon}(x)$ where

$$\varphi_i^{\varepsilon}(x) = \left\{ x_i^* \in \Delta(S_i) : x_{ih}^* \leq \varepsilon x_{ik}^* \text{ if } \tilde{\pi}_i\left(e_i^h, x_{-i}\right) < \tilde{\pi}_i\left(e_i^k, x_{-i}\right) \right\}$$

3. Convex-valued, compact-valued and u.h.c. Apply Kakutani to φ^{ε}

4. Each fixed point under φ^{ε} is ε-proper

Example 1.5 *Reconsider the expanded entry-deterrence game, with a "dumb" strategy. We saw that* (A, F) *is perfect. But is it proper? No, because mistake D is more costly than mistake E.*

Remark 1.1 *van Damme showed that also the set of proper equilibria can change when one adds a strictly dominated strategy to a game. However, van Damme also showed that properness has a remarkable and beautiful implication for extensive-form analysis (later lecture!)*

1.3 Essentiality

- Discard NE that are not robust to perturbations of *payoffs*!

- The players know the payoffs, but the analyst is not absolutely sure

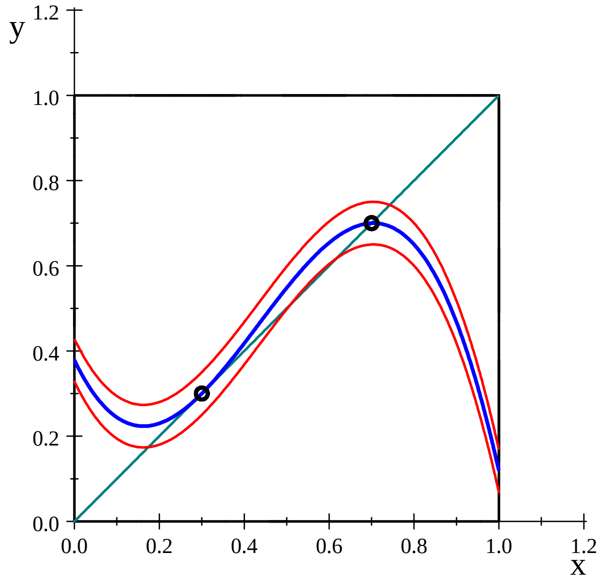
- Given $G = (N, S, \pi)$, $\pi : S \to \mathbb{R}^n$, consider all games $G^* = (N, S, \pi^*)$, for $\pi^* : S \to \mathbb{R}^n$

- Define distance between games G and G^*:

$$d(G, G^*) = \max_{i \in N, s \in S} \mid \pi_i(s) - \pi_i^*(s) \mid .$$

Definition 1.4 (Wu and Jiang, 1962) $x \in \square^{NE}$ *is* **essential** *if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every game G^* within payoff distance δ from G has a Nash equilibrium within distance ε from x.*

- Based on Fort (1950), "essential fixed points under continuous mappings"



Proposition 1.6 *Essentiality ⇒ Perfection*

Proof idea: Trembles can be viewed as special forms of payoff perturbations.

However:

- Does still not reject the mixed equilibrium in

	L	R
T	2, 2	0, 0
B	0, 0	1, 1

- and some games have no essential equilibria at all:

	L	R
T	1, 2	0, 0
B	1, 3	0, 0

2 Set-valued solutions

- The EF generic constancy of outcomes on NE components (Kreps and Wilson, 1982)

2.1 Essential NE components

Definition 2.1 (Jiang, 1963) *A non-empty closed set* $X \subset \square^{NE}$ *is* **essential** *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that every game* G^* *within payoff distance* δ *from* G *has a Nash equilibrium within distance* ε *from* X*.*

Proposition 2.1 *Let* G *be a finite game. At least one component of the set* $\square^{NE}$ *is essential.*

Proposition 2.2 *Let G be a finite game. Every essential component of $\square^{NE}$ contains at least one strategically stable set.*

- However, we still do not get rid of the mixed NE in the coordination game

	L	R
T	2, 2	0, 0
B	0, 0	1, 1

2.2 Strategic stability

- Elon Kohlberg and Jean-Francois Mertens (1986) showed that any point-valued solution concept, no matter how defined, will have some theoretical flaw.

- They therefore suggested that we instead define solutions set-wise.

- Instead of considering a strategy profile $x \in \square(S)$ consider a (closed) **subset** $X \subset \square(S)$

Definition 2.2 (Kohlberg-Mertens, 1986) $X \subset \square(S)$ *is* **strategically stable** *if it is minimal with respect to the following property: X is non-empty and closed, and for every $\varepsilon > 0$ there is some $\delta > 0$ such that every strategy-perturbed game $G(\lambda) = (N, X(\lambda), u)$ with mistake probabilities $\lambda_{ih} < \delta$ has some Nash equilibrium within distance ε from X.*

- Minimality means that the set has no proper subset with the stated property

- One can show that if X is strategically stable, then $X \subset \Box^{PE}$.

- However, we still do not get rid of the mixed NE in the coordination game

- And, also disturbing is that strategically stable sets may contain equilibria from different NE components

- For a fix of this and other theoretical flaws, see Mertens (1989)

- Next lecture: Solution concepts for finite EF games, and relations to NF solutions

THE END