Introduction to Game Theory
Problem Set #1

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1. Consider the following 2×2 normal-form game G, for arbitrary $a, b > 0$:

	H	T
H	$a, 0$	$0, b$
T	$0, b$	$1, 0$

   (i) Find all pure and mixed Nash equilibria in G.

   (ii) For each pure or mixed Nash equilibrium in G, and each player, find the player's set of pure and mixed best replies to the equilibrium in question.

2. Two individuals, 1 and 2, contribute to a public good (say, a clean shared kitchen) by making individual efforts $x \in [0, 1]$ and $y \in [0, 1]$. The resulting level of the public good is $x + y$. Individual utilities are given by

$$u_1(x, y) = (x + y)\, e^{-x} \quad \textit{and} \quad u_2(x, y) = (x + y)\, e^{-y}$$

   Each individual strives to maximize his or her expected utility.

   (a) Game A: Suppose both effort levels are chosen simultaneously. Write up the normal form of Game A! Is this a Euclidean game? Does it have a pure-strategy Nash equilibrium? Find the set of pure-strategy Nash equilibria!

   (b) Game B: Suppose individual 1 first chooses her effort level, and that this is observed by individual 2, who then chooses his effort level. Write up the normal form of Game B! Is it a Euclidean game? Does it have a pure-strategy Nash equilibrium? Find its unique subgame perfect equilibrium in pure strategies!

   (c) Does there exist a Nash equilibrium in Game B in which individual 2 makes effort $x_2 = 1/2$? (Either prove than none exists or specify one such equilibrium.)

3. Consider an n-player simultaneous game, with normal form G, in which each player has k pure strategies, which we below will call *actions*.

   (i) How many pure-strategy profiles does G have? How does this number grow if k is doubled, if n is doubled?

   (ii) Suppose that the game is played twice, and that all players, before chosing their second-period action, observes the first-period action profile, and hence can condition their second-period action upon this observation. Let G^2 be the normal form of the so-defined game. How many pure strategies does each player have in G^2 when $n = k = 2$? When $n = 2$ and $k = 10$?

(iii) Is it true that the game G^2 necessarily has at least one Nash equilibrium in pure strategies, for any possible assignment of Bernoulli values?

(iv) Is it true that the game G^2 necessarily has at least one Nash equilibrium in mixed strategies, for any possible assignment of Bernoulli values?

4. Draw the extensive form for a strategic interaction in which player 1 has a binary choice, to take action A or B, and where the interaction ends if 1 takes action A but continues if he takes action B. In the latter case, player 2 learns that 1 took action B, and player 2 then has a binary choice, to take action C or D. If player 1 ends the game by taking action A, then each player receives 5 euros. If player 2 ends the game by taking action C then player 1 receives 6 euros and player 2 receives 1 euro, while if player 2 ends the game by taking action D then both players receive nothing.

   (a) Suppose that each player's payoff (that he or she strives to maximize) is his or her monetary gain, and solve the so-defined game by backward induction.

   (b) Suppose instead that player 1 is an altruist in the sense that her payoff is the sum of both player's monetary gains, but player 2 is selfish and only care about his own gain. Solve the so-defined game by backward induction.

   (c) Suppose instead that player 1 is selfish and only cares about her own monetary gain, while player 2 is spiteful against people who take action B (1 thinks of such persons as greedy). More exactly, 1's payoffs are as case (a) above, while 2's payoff is 5 if player 1 takes action A, but $x - y$ if player 1 takes action B, where x is 2's monetary gain and y that of player 1. Compare the equilibrium behavior of the selfish player 1 with that of the altruistic player 1 in (b).

5. Consider n firms competing in a product market with demand $D(p) = \max\{0, a - p\}$, for some $a > 0$. In stage 1 of the interaction, each firm i selects its output level $q_i \in [0, 100]$, without observing other firms' outputs. Let $Q = q_1 + ... + q_n$. In stage 2, the market price is determined by the equation

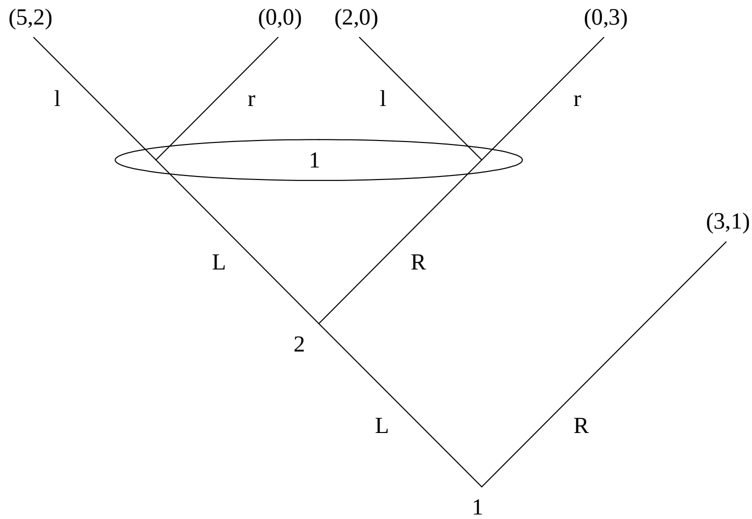
$$D(p) = Q = q_1 + ... + q_n$$

   Suppose that the profit to firm i is $\pi_i = (p - c_i) q_i$, for some $c_i < a$. Suppos that each firm strives to maximize its profit.

   (a) Let $n = 2$ and suppose that both firms' managers are rational and that their rationality and all the details of the game (including their costs) are common knowledge between them (each knows this and knows that the other knows this etc. *ad infinitum*). Show that this uniquely determines their output levels, and that this coincides with the unique Nash equilibrium of the game.

   (c) Analyze the questions in (a) but now for an arbitrary finite number n of firms, and study how the outcome depends on n. In particular, what happens as $n \to \infty$?

6. Consider the extensive-form game Γ given below.



(a) Verify that Γ has perfect recall and that it has two subgames.

(b) Write up its pure-strategy normal form, as well as its quasi-reduced, reduced normal form, and agent normal form.

(c) Prove directly that the claim in Kuhn's Theorem holds in this game. (That is, that there for each mixed strategy in the normal form of the game exists a realization-equivalent behavior strategy.

(d) Find the set of pure-strategy Nash equilibria.

7. Consider a two-player finite extensive-form game of perfect information, Γ_T, defined as follows: At each decision node, the player in question has only two choice alternatives, or *moves*, to go L or R. The players take turns: first player 1 chooses between L and R, then 2 observes 1's choice and chooses L or R, then 1 observes 2's choice and makes his move, etc. until both players have made T moves each. The game thus has "length" $2T$, where T is a positive integer.

(i) How many pure strategies does each of the players have in this game when $T = 2$? When $T = 3$?

(ii) For arbitrary $T \in \mathbb{N}$?

(iii) Let $T \in \mathbb{N}$ and suppose that the Bernoulli values in Γ_T have been drawn at random, i.i.d. and normally distributed. After the realization of this draw, both players know all Bernoulli values. What is the probability, from the perspective of an outside observer, that the so created game has at least one Nash equilibrium in pure strategies? What is the probability that the game has at least one subgame perfect equilibrium in pure strategies?