Introduction to Game Theory
Problem Set #4

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1. Consider infinitely repeated play of the two-player simultaneous-move game with payoff bi-matrix

	a	b	c
a	$3,0$	$4,4$	$1,0$
b	$2,2$	$0,0$	$0,0$
c	$0,1$	$0,1$	$0,0$

(a) Identify its pure-strategy *minmax* strategies and payoffs, and its set of *feasible payoff pairs*. Show all this in a diagram.

(b) What does the Fudenberg-Maskin Folk Theorem for infinitely repeated games with discounting say about this example?

(c) Suppose both players' discount factor is $\delta = 2/3$. Find a subgame perfect strategy profile that supports the play of the action pair (b, a) in all periods, and hence the average discounted payoff $(1-\delta)\sum_t \delta^t \pi_i(t) = 2$ to each player. Specify the strategies (that is, what action each player will take after every possible history of play).

(d) Suppose the stage game in (c) is repeated a *finite* number of times. Find its average discounted subgame perfect equilibrium payoffs, for any $\delta \in (0, 1)$. Compare this with your findings in (c), and explain the difference.

2. Consider a two-player simultaneous-move game G with normal form

	L	M	R
A	$10,13$	$-5,0$	$-5,0$
B	$11,-5$	$4,1$	$0,0$
C	$0,-5$	$0,0$	$1,4$

(a) Find all *strictly dominated* pure strategies.

(b) Find all Nash equilibria (in pure and mixed strategies). Compute the (expected) payoff pair (π_1, π_2) for every Nash equilibrium, and place these points in a diagram, with the first player's payoff on the horizontal axis and the second player's payoff on the vertical.

(c) Find a (pure or mixed) *minmax strategy* against each player, compute that player's *minmax payoff*, and place the corresponding point in the diagram drawn in (b).

(d) Define the set of feasible and individually rational payoff pairs and indicate it in the diagram.

(e) Suppose this game G is repeated twice, where each player's payoff in the repeated game is the *average* of his or her payoffs in each period, and where both players observe each others' first-period actions before deciding on their second-period actions. Can the (average) payoff pair $(7, 7)$ be obtained in subgame perfect equilibrium in the repeated game?

(f) Suppose that the game G is repeated infinitely many times, with perfect monitoring between periods, and where both players discount future payoffs by the same factor δ. For what values of $\delta \in (0, 1)$ can the average discounted payoff pair $(10, 13)$ be obtained in pure-strategy subgame perfect equilibrium of the infinitely repeated game? Specify the strategies (that is, what action each player will take after every possible history of play).

3. Consider a Cournot oligopoly game between firms $i = 1, 2, ..., n$, where demand is linear and the marginal production cost is constant and equal for both firms. Let the inverse demand function be $P(Q) = \max\{0, 1 - Q\}$, where $Q = q_1 + ... + q_n$ is aggregate output, and let the marginal cost be c for all firms, where $0 < c < 1$. Hence, each firm's profit is $\pi_i(q_1, ..., q_n) = [P(Q) - c] \cdot q_i$

   (a) Define the described situation as a simultaneous-move game in normal form, with profits as payoffs, and show that it has a unique (pure strategy) Nash equilibrium. Find this, and compute the corresponding profits.

   (b) Suppose this interaction is repeated over time, in periods $t = 0, 1, 2, 3, ...$ Suppose that all firms use pure behavior strategies and can observe each other's outputs after each period. After each period there is a probability $p \in (0, 1)$ that the interaction stops, with statistical independence between periods. The payoff to each firm is its expected total profit. Define this as an infinitely repeated game with discounting, and identify the discount factor δ as a function of p.

   (c) In the setting in (b) and for $n = 2$: What exactly does the Fudenberg-Maskin Folk Theorem for subgame perfect equilibrium say in this example? Give an example of a subgame-perfect equilibrium, with a stationary outcome $\left(q_1^0, q_2^0\right)$, each period, in which firm 1 earns more than in the static Nash equilibrium but firm 2 earns less than its static Nash equilibrium payoff, supported by temporary minmaxing, and specify the range of δ-values for which your strategy profile is subgame perfect.

   (d) In the setting in (b) and for $n > 2$: Does the Abreu-Dutta-Smith Folk Theorem apply? What does it say in this example? Consider subgame perfect equilibrium strategies whereby the n firms each produce $1/n$:th of the monopoly output quantity, Q^m, supported by the threat of Nash-reversion, that is, when any deviation is punished by play of the static Nash equilibrium forever. For each $n \geq 2$, find the minimal discount factor δ for which this is possible, in particular, how it depends on n and its asymptotic value as $n \to \infty$.

4. Consider two altruistic players who each contributes to a public good. For any contributions $x_1, x_2 \geq 0$ let

$$u_i\left(x_1, x_2\right) = \left(x_1 + x_2\right)^\tau - \frac{1}{2}x_i^2$$

be the resulting (material) utility to player i, where $0 < \tau \leq 1$. Each player i places some positive weight $\alpha_i \in (0, 1)$ on the other player's utility, so that player i's (psychological) utility is

$$\pi_i\left(x_1, x_2\right) = u_i\left(x_1, x_2\right) + \alpha_i u_j\left(x_1, x_2\right)$$

for $j \neq i$. View this as a simultaneous-move game with payoff functions π_i.

(a) Find the unique Nash equilibrium (x_1^*, x_2^*) of this game, and calculate the associated *material* utilities.

(b) For the special case $\alpha_1 = \alpha_2 = \alpha \in (0, 1)$, find the action pair (x_1^o, x_2^o) that maximizes the sum of the players' psychological utilities, and compare with the Nash equilibrium action pair for this special case. Does (x_1^o, x_2^o) depend on α?

(c) For the special case $\alpha_1 = \alpha_2 = \alpha \in (0, 1)$, and $\tau = 1$, suppose that this game is repeated indefinitely, and that both players discount future (psychological) utility by the same discount factor $\delta \in (0, 1)$. For each such discount factor, identify the set of action pairs (x_1, x_2) such that perpetual play of this action pair is the outcome of a subgame perfect equilibrium in which each player punishes any deviation by reversion forever to the static Nash equilibrium. For what range of altruism coefficients and discount factors, $\alpha, \delta \in (0, 1)$, is (x_1^o, x_2^o) implementable in this way?