COURSE ESSAY FOR THE COURSE AN INTRODUCTION TO GAME THEORY – SPRING 2010

JONAS WESTIN

SUMMARY OF THE PAPER *EVOLUTIONARY STABILITY OF PROSPECT THEORY PREFERENCES* BY MARC OLIVER RIEGER (2009)

INTRODUCTION

The paper studies the effect of prospect theory preferences on the outcome of two player games. The main focus is on the effect of applying probability weighting of the opponents mixed strategy probabilities and comparing these with the outcome given by expected utility theory in an evolutionary game theory setting (Rieger, 2009).

Since the prospect theory probability weighting function overweight small probabilities this leads to suboptimal decisions (from an expected utility point of view). Using these seemingly irrational preferences will therefore likely reduce the expected payoff in a situation where the probabilities are exogenously given or static. When interactions between individuals are taken into account, these non-rational preferences can however become profitable and evolutionary stable as is shown in this paper.

The paper examines a class of 2x2 games called social control games similar to the matching pennies games and demonstrates that prospect theory preferences can be evolutionary stable, i.e. a population with prospect theory preferences earns on average more than a small fraction of (rational) expected utility maximizing intruders, even when the actual received payoff is given by expected utility theory.

A SHORT INTRODUCTION TO PROSPECT THEORY

In the standard formulation of game theory, the payoff from a mixed strategy is given by expected utility theory, i.e. the value of a game is given by $u(x,p) = \sum_{i=1}^{N} p_i x_i$ where p_i is the probability that outcome x_i occurs.

The actual behavior of individuals making decisions under risk does however often deviate from what is rational according to expected utility theory. Prospect theory, developed by Kahneman and Tversky (1979), is one model that tries to explain some of the inconsistencies between the observed and the “rational” decisions under risk. Prospect theory modifies expected utility theory so that when making a choice between several alternatives; only the gains and losses matters and not the final wealth; a loss is perceived as worse than a gain; and a small probability is overweighted and a probable to large probability is underweighted.

These features can be expressed mathematically by the following two transformations of the utility x and the probability distribution p. The parametric functions are from Tversky and Kahneman (1992). First the utility function is replaced by a S-shaped value function which is concave in gains and convex in losses,

$$u(x) = \begin{cases} x^{\alpha} & x \geq 0 \\ -\lambda(-x)^{\alpha} & x < 0 \end{cases} \tag{1}$$

where $\alpha, \beta \in (0,1)$ and $\lambda > 1$. The probability is transformed with a weighting function w as follows

$$\mathrm{w}_{\gamma}(\mathrm{p}) = \frac{\mathrm{p}^{\gamma}}{(\mathrm{p}^{\gamma}+(1-\mathrm{p})^{\gamma})^{1/\gamma}}. \tag{2}$$

The subjective utility of a choice with outcomes x_i with probabilities p_i are

$$\mathrm{PT}(\mathrm{x},\mathrm{p}) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{w}_{\gamma}(\mathrm{p_i})\mathrm{u}(\mathrm{x_i}). \tag{3}$$

PROSPECT THEORY PREFERENCES IN GAMES

The paper discusses both the classical prospect theory and a newer version called cumulative prospect theory. The paper uses a probability transformation of the opponents mixed probabilities to incorporate some of the features of prospect theory into game theory. The full paper contains several generic games. In this essay we will only look at a single numerical example to demonstrate the basic concept behind the theory. To illustrate the theory we will consider a simple 2x2 game from the paper.

		Player B L	Player B R
Player A	T	(4,2)	(-2,3)
	B	(3,2)	(0,0)

This game has a mixed unique Nash equilibrium given by $(p,q) = \left({}^{2}/_{3}, {}^{2}/_{3}\right)$ where p is the probability that player A chooses top and q is the probability that player B chooses left. Using the probability weighting function (2) with the parameters $\alpha \in (0,1]$ and $\beta \in (0,1]$ for player A and B respectively, we get the PT-utilities for the players as,

$$\begin{aligned} U_A(q,p) &= 4pw_\alpha(q) - 2pw_\alpha(1-q) + 3(1-p)w_\alpha(q) \\ U_B(p,q) &= 2w_\beta(p)q + 2w_\beta(1-p)q + 3w_\beta(p)(1-q) \end{aligned} \quad (4)$$

Note that we assume that the individual uses the weighed probability for the other player's choice but the non-weighed probability for his or her own choice. From the PT-utilities in equation (4) we can now calculate the unique mixed Nash equilibrium

$$(p,q) = \left(\frac{2^{1/\beta}}{1+2^{1/\beta}}, \frac{2^{1/\alpha}}{1+2^{1/\alpha}}\right). \quad (5)$$

We see that the equilibrium approaches the non-weighted equilibrium when the weight parameter goes to one. This can also be seen from the fact that the PT-utilities collapses into the expected utility theory utilities when $\alpha \to 1$ and $\beta \to 1$. We also see that the probabilities are strictly decreasing in α and β, i.e. if $\beta > \alpha$ then $p_\beta < p_\alpha$.

We will now study the evolutionary stability of the game above. Consider a situation following the Nash mass-action interpretation where the game above is embedded in a larger game where individuals in a population are randomly called upon to play the game. The individual has an equal probability to play as player A or player B. All the individuals in the population have PT-utilities but can have different weight parameters $\gamma \in (0,1]$.

The probability weighting are only used as a behavioral bias of the players' decisions and is not reflected in actual payoff received from the game. This means that the growth of a sub-population (in an evolutionary stability sense) with weight parameter α is given by the expected utility with the real probabilities and not the weighted ones.

We call an individual with a probability weight $\alpha \in (0,1]$ a α-weighter. The average rational utility an α-weighter obtains against a β-weighter is $U(\alpha,\beta) = \frac{1}{2}\left(U_A(\alpha,\beta) + U_B(\beta,\alpha)\right)$, where U_A and U_B is the expected utility for playing as player A and B respectively.

A probability weighting $\alpha \in (0,1]$ is called evolutionary semi-stable[1] if for all $\beta \in (0,1]$ with $\alpha < \beta$ and for all sufficiently small $\varepsilon > 0$ the expected rational utility of the α-weighters is larger than the rational utility of the β-weighters, where the proportions of the two sub-populations of α-weighters and β-weighters are $1-\varepsilon$ and ε respectively, i.e.

$$\varepsilon U(\alpha,\beta) + (1-\varepsilon)U(\alpha,\alpha) > \varepsilon U(\beta,\beta) + (1-\varepsilon)U(\beta,\alpha). \qquad (6)$$

The game in the example is evolutionary semi-stable for sufficiently, i.e. for sufficiently large $\alpha < 1$ a population of α-weighters cannot be invaded by β-weighters with $\beta > \alpha$, i.e. individuals with a smaller behavioral bias.

Let $\beta > \alpha$ and look at the inequality (6).

$$\Delta(\varepsilon) = \varepsilon U(\alpha,\beta) + (1-\varepsilon)U(\alpha,\alpha) - \varepsilon U(\beta,\beta) - (1-\varepsilon)U(\beta,\alpha) \qquad (7)$$

Let p_β and q_α be the PT Nash equilibrium strategies if player A is an α-weighter and player B a β-weighter and so on. First we prove that the inequality holds for $\varepsilon = 0$.

$$\begin{gathered}\Delta(0) = U(\alpha,\alpha) - U(\beta,\alpha) = \\ \tfrac{1}{2}\Big(\big(U_A(\alpha,\alpha) + U_B(\alpha,\alpha)\big) - \big(U_A(\beta,\alpha) + U_B(\alpha,\beta)\big)\Big) = \\ \tfrac{3}{2}\Big(\big(q_\alpha - q_\beta\big)(1+p_\alpha) + \big(p_\alpha - p_\beta\big)(1-q_\alpha)\Big)\end{gathered} \qquad (8)$$

Since p and q are strictly decreasing functions in α and β we have that $p_\alpha > p_\beta$ and hence $\Delta_0 > 0$. Since the function $\Delta(\varepsilon)$ is continuous in ε we deduce that the inequality also holds for sufficiently small values of $\varepsilon > 0$.

Discussion

What can we learn from this example; even though the probability weighting is a behavioral bias that usually leads to suboptimal decisions, it can be evolutionary stable in certain games. It can thus sometimes be optimal to have prospect theory preferences rather than expected utility preferences.

The paper only deals with the probability weighting part of Prospect theory, what would be interesting, although more complicated, is to also include an analysis of how gains and losses in repeated or sequential games can affect the players behavior compared to a standard analysis where only the final outcome matters.

References

Rieger, M.O. (2009), Evolutionary Stability of Prospect Theory Preferences, Working Papers, Bielefeld University, Institute of Mathematical Economics

Kahneman, D. and Tversky, A. (1979), Prospect Theory: An analysis of decision under risk, Econometrica 47, 263-291.

[1] If (6) also holds for $\alpha > \beta$ the probability weighting α is called evolutionary stable.