Summary of the article "Evolutionary Stability of Portfolio Rules in Incomplete Markets"

Ali Hamdi

May 25, 2010

1 Introduction

The article treats portfolio rules in financial markets. More specifically, the authors introduce some stability concepts, and derive conditions for the investing strategies to be stable according to these concepts. In a special case, when the assets only have binary payoffs, the optimal portfolio rules are derived explicitly by means of maximization of a particular expected value. It shown that the result of this is a portfolio rule which asymptotically dominates the market (still in the case of binary payoffs).

After this, general payoffs in incomplete markets are considered. In three different cases of process used to model the state of nature, the explicit rules which are evolutionary stable (according to the definition introduced in the article), is derived. It is also shown that in the particular case of binary payoffs, this reduces to the same rules that were derived earlier by means of finding the asymptotically dominant strategy.

Finally, an application to mean-variance and CAPM portfolio rules is treated. It is shown that mean-variance will, due to under-diversification, yield portfolios that are not evolutionary stable, while the CAPM rule will give portfolios which are resistant to the market selection mechanism. I.e., it is a strategy which will not be driven out of the market.

2 Definitions

The following is a list of definitions that constitutes the set-up of the paper.

- t - Time. Discrete and thereby also serves as index.
- Ω - Path space of the underlying stochastic process. Each element of Ω can be written as $\omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots)$, where ω_t is the state of nature at time t.
- $\mathcal{F}$ - Sigma algebra that corresponds to Ω.
- $\mathbb{P}$ - Probability measure on $\mathcal{F}$.

- ω^t - The sequence of observations of the state of nature up to the end of time t.
- $\mathcal{F}^t$ - Information set that corresponds to ω^t. I.e., $\mathcal{F}^t = \sigma \{\omega_u : \ u \leq t\}$.
- θ - The shift operator defined by the relation

$$\theta\omega_t = \omega_{t+1}, \quad \forall t.$$

  In the same way, θ^t is the shift operator applied t times and θ^0 is the identity operation.
- i - Index of the investors. $i = 1, \ldots, I$. I.e., there are I investors.
- k - Index of the assets. $k = 1, \ldots, K$ where $K \geq 2$, i.e. there are at least two assets.
- $A_t^k(\omega)$ - Payoff of asset k in period t, given the state of nature ω.
- a_t^i - The portfolio of investor i in period t. It is a mapping from Ω to $\mathbb{R}_+^K$ and can hence be represented as $a_t^i = \left(a_{1,t}^i, \ldots, a_{K,t}^i\right)$, where $a_{1,t}^i : \ \Omega \to \mathbb{R}$ is the investors holdings in absolute numbers in asset k.
- w_t^i - The wealth of investor i at the end of period t. $w_0^i > 0$ by assumption, and the above yields that

$$w_{t+1}^i = \sum_{k=1}^{K} A_{t+1}^k(\omega)\, a_{k,t}^i.$$

- S_k^t - Total exogenous supply of asset k at time t.
- $\rho_{k,t}$ - The price of asset k in period t. Prices are assumed to be determined endogenously by the equilibrium of supply and demand. Thus, the prices can be written as

$$\rho_{k,t} = \frac{1}{S_k^t} \sum_{i=1}^{I} \lambda_{k,t}^i w_t^i.$$

- $\lambda_{k,t}^i$ - The budget share of investor i in asset k in period t. I.e., it is the fraction of the investors total wealth that is invested in asset k. Provided that the wealth is positive, $\lambda_{k,t}^i$ is hence defined as

$$\lambda_{k,t}^i = \frac{\rho_{k,t} a_{k,t}^i}{w_t^i}.$$

- W_t - Total market wealth at time t, i.e.

$$W_t = \sum_{i=1}^{I} w_t^i.$$

By some manipulations, the following is shown to hold

$$W_t = \sum_{i=1}^{I} w_t^i = \sum_{k=1}^{K} A_t^k(\omega) S_{t-1}^k.$$

- $q_{k,t}$ - Normalized market prices, defined by

$$q_{k,t} = \frac{\rho_{k,t}}{W_t}.$$

- $R_t^k(\omega)$ - Relative payoff of asset k, defined by

$$R_t^k(\omega) = \frac{A_{t+1}^k(\omega) S_t^k}{\sum_{l=1}^{K} A_{t+1}^l(\omega) S_t^l}.$$

  These are assumed to be stationary.

- r_t^i - Market shares of investor i, in time t. The basic definition is

$$r_t^i = \frac{w_t^i}{W_t},$$

  but after some manipulations, the following recursive formula is shown to hold:

$$r_{t+1}^i = \sum_{k=1}^{K} R_t^k(\omega) \frac{\lambda_{k,t}^i r_t^i}{\sum_{j=1}^{I} \lambda_{k,t}^i r_t^i},$$

  and so the evolution of market shares can be written as

$$r_{t+1} = f\left(\theta^{t+1}\omega, r_t\right),$$

  where

$$f_i\left(\theta^{t+1}\omega, r\right) = \sum_{k=1}^{K} R^k\left(\theta^{t+1}\omega\right) \frac{\lambda_k^i(\theta^t\omega) r^i}{\sum_{j=1}^{I} \lambda_k^j(\theta^t\omega) r^j}.$$

  This is referred to as the market selection process.

- $\varphi(t, \omega, r)$ - The vector of wealth shares of all investors at time t, given an initial distribution of wealth r, and given that the sequence of states ω is observed. I.e., it holds that

$$\varphi(t, \omega, r) = f\left(\theta^t\omega\right) \circ \ldots \circ f(\theta\omega) \circ r.$$

  The family of these maps forms what is called a random dynamical system on the unit simplex Δ^I.

3 Stability concepts

In this section, the relevant stability concepts are defined.

Definition 1. A distribution r of market shares is said to be invariant under the market selection process if for all $\omega \in \Omega$ and all t it holds that

$$r = \varphi(t, \omega, r).$$

Thus, a distribution r is said to be invariant if it is a fixed point of the mapping φ.

Definition 2. An invariant distribution of market shares $\overline{r} \in \Delta^I$ is stable if

$$\lim_{t\to\infty} \|\varphi(t, \omega, r) - \overline{r}\| = 0,$$

for all r in a neighborhood of $\overline{r}$ for all ω. This neighborhood may depend on ω.

In other words, if the starting distribution is instead of $\overline{r}$ chosen to be r, but still in some neighborhood of $\overline{r}$, and the resulting distribution is asymptotically the same, then $\overline{r}$ is said to be a stable distribution. The following definition extends this concept, but to the case when there may be new investors entering the market.

Definition 3. Consider the case when there is I investors in the market, and let $\mathcal{I} = \{1, \dots, I\}$ be the corresponding set of indices. Assume that there are J new investors entering the market, and let $\mathcal{J} = \{1, \dots, J\}$ be their corresponding indices. Then and invariant distribution of market shares $\overline{r} \in \Delta^I$ is said to be evolutionary stable if for all $J \geq 0$, $(\overline{r}, 0, \dots, 0) \in \Delta^{I+J}$ is stable for the random dynamical system with trading strategies $\left(\left(\lambda^i\right)_{i\in\mathcal{I}}, \left(\lambda^i\right)_{i\in\mathcal{J}}\right)$.

A trading strategy is called evolutionary stable, if the invariant distribution of market shares $1 \in \Delta^1$ is evolutionary stable.

Lastly, a local stability criterion, taking into account possible entry barriers, is defined below.

Definition 4. An invariant distribution of market shares $\overline{r} \in \Delta^I$ is called locally evolutionary stable, if for all $J \geq 0$ there exists a random variable $\delta(\omega) > 0$ such that $(\overline{r}, 0, \dots, 0) \in \Delta^{I+J}$ is stable for all sets of portfolio rules $\left(\left(\lambda^i\right)_{i\in\mathcal{I}}, \left(\lambda^i\right)_{i\in\mathcal{J}}\right)$ with

$$\min_{i\in\mathcal{I}} \max_{j\in\mathcal{J}} \left\|\lambda^i(\omega) - \lambda^j(\omega)\right\| < \delta(\omega)$$

for all ω.

A portfolio rule is called locally evolutionary stable, if the invariant distribution of market shares $1 \in \Delta^1$ is locally evolutionary stable.

4 Assumptions

The following assumptions are made in the article.

1. Prices are determined endogenously by the equilibrium of supply and demand.
2. All assets yield non-negative payoffs in all states of nature, each asset has strictly positive payoffs for a set of states with positive measure, and the total payoff of all assets is strictly positive in every state.
3. In every market there is some trading strategy λ_t^i with initial wealth $w_0^i > 0$ that is completely mixed, i.e. $\lambda_t^i(\omega) \in \mathrm{int}\Delta^K$ for all $\omega \in \Omega$.
4. The relative payoffs are stationary random variables.
5. Payoffs are assumed to be binary (at first): $R^k(\omega) \in \{0,1\}$ with $R^k(\omega) = 1$ if and only if $\omega_0 \in S_k$, where $(S_k)_{k=1,\ldots,K}$ is some partition of S.
6. There are no redundant assets in the sense that different portfolios do not yield the same payoff almost surely.

5 Optimal portfolio choice assuming binary payoffs

Under the assumption of digital payoffs, the evolution of market shares can be simplified to

$$r_{t+1}^i = \frac{\lambda_{\omega_{t+1}}^i r_t^i}{\sum_{j=1}^I \lambda_{\omega_{t+1}}^j r_t^j},$$

where $\lambda_{\omega_{t+1}}^i = \sum_k R^k(\omega_{t+1})\lambda_k^i(\omega^t)$ is the wealth share invested in asset k by investor i, with $\omega_{t+1} \in S_k$. And so the evolution of the ratio of market shares between any two investors can be written as

$$\frac{r_{t+1}^i}{r_{t+1}^j} = \frac{\lambda_{\omega_{t+1}}^i}{\lambda_{\omega_{t+1}}^j}\frac{r_t^i}{r_t^j}.$$

From this, it is shown in the article that the investors that are closest to maximizing the expected logarithm of the wealth shares will eventually dominate the market. I.e., among all adapted strategies, the optimal portfolio rule is the random variable $\lambda(\omega) = \lambda\left(\omega^0\right) \in \Delta^K$ that maximizes

$$\mathbb{E}\left[\ln\left(\sum_{k=1}^K R^k(\theta\omega)\lambda_k(\omega)\right)|\mathcal{F}_0\right] = \mathbb{E}\left[\ln\left(\lambda_{\omega_1}\right)|\mathcal{F}_0\right],$$

where $\lambda_{\omega_1} = \lambda_k\left(\omega^0\right)$ if and only if $\omega_1 \in S_k$.

In the article, two particular cases are considered. First, the state of nature is assumed to follow an i.i.d. process. Then, since the payoffs $R^k(\theta\omega)$ are independent of the past $\mathcal{F}_0$, the expression to be maximized reduces to

$$\sum_{k=1}^{K} p_k \ln\left(\lambda_k^i\right),$$

from which it follows that the optimal choice is $\lambda_k^* = p_k$.

Secondly, the state of nature is assumed to follow a Markov process with time-homogeneous transition probabilities $P(\omega_{t+1}, \omega_t)$. Then the expression to be maximized can be written as

$$\sum_{k=1}^{K} P(S_k, \omega_0) \ln\left(\lambda_k(\omega)\right),$$

and so the optimal choice is $\lambda_k^* = P(S_k, \omega_0)$. Furthermore, the stationarity yields that $\lambda_k^*(\theta^t\omega) = P(S_k, \omega_t)$.

6 Main results

In this section, the main results of the article is presented. Here, the assumption of binary payoffs is relaxed and general payoffs in an incomplete market are considered. However, we are still restricted to invariant deterministic distributions of wealth shares. Firstly, a proposition that characterizes all deterministic invariant distributions of wealth shares:

Proposition 1. Only one portfolio rule can have strictly positive wealth in every population of strategies with a (deterministic) invariant distribution of wealth shares.

Secondly, a proposition that states which portfolio rules that are stable, given certain conditions, is presented.

Proposition 2. Let the state of nature be determined by an ergodic process. Given any set of adapted portfolio rules (λ^i). The invariant distribution of market shares $\overline{r} = e_n$ being concentrated on the players of the completely mixed n-th strategy is

stable, if

$$\mathbb{E}\left[\ln\left(\sum_{k=1}^{K} R^k(\theta\omega) \frac{\lambda_k^i(\omega)}{\lambda_k^n(\omega)}\right)\right] < 0, \quad \forall i \neq n,$$

unstable, if

$$\mathbb{E}\left[\ln\left(\sum_{k=1}^{K} R^k(\theta\omega) \frac{\lambda_k^i(\omega)}{\lambda_k^n(\omega)}\right)\right] > 0,$$

for some $i \neq n$.

From this it follows that there is a systematic way of checking stability of portfolio rules, namely checking the results of Proposition 2. The following results of the paper is based on this.

After this, the authors derive a Corollary that in essence, given certain conditions, there can only be one evolutionary stable portfolio rule:

Corollary 1. Suppose all investors employ completely mixed portfolio rules. Then existence of a portfolio rule that is evolutionary stable implies that all other portfolio rules cannot be evolutionary stable.

In the light of this, what the authors did, was to show the existence of this unique evolutionary stable portfolio rule and state an explicit formula for this rule.

For future reference, we state the definition of an ergodic process:

Definition 5. A stochastic process is said to be ergodic if its statistical properties (such as mean and variance) can be obtained from a realization of the process, given that the sample is sufficiently large.

The following three results are the main results of the paper. These all give the explicit portfolio rule that is evolutionary stable, but under different conditions. The first theorem assumes only the state of nature to be an ergodic process, the second assumes it to be an i.i.d. process, and the third theorem assumes it to be a Markov process.

Theorem 1. Let the state of nature be determined by an ergodic process. Suppose investors only employ simple strategies, i.e. $\lambda(\omega) \equiv \lambda \in \Delta^K$. Then the simple strategy λ^* defined by,

$$\lambda^* = \mathbb{E}R^k(\omega)$$

for $k = 1, \ldots, K$ is evolutionary stable, and no other strategy is locally evolutionary stable.

It should be noted that in the case of binary payoffs, this reduces to the same result as the one derived earlier.

Theorem 2. Let the state of nature be determined by an i.i.d. process. Then $\lambda_k^* = \mathbb{E}R^k$, $k = 1, \ldots, K$, is the only evolutionary stable portfolio rule.

Moreover, if $\mathcal{S}$ is the power set of the set of states S, then we find that all other completely mixed adapted strategies are not even locally evolutionary stable.

Theorem 3. Let the state of nature be determined by a Markov process (with transition probability P). Then the adaptive strategy λ^* defined by,

$$\lambda_k^*(\omega_0) = \mathbb{E}\left[R^k(\omega_0)\,|\omega_0\right] = \int_S R^k(s)\,P(ds, \omega_0),$$

for $k = 1, \ldots, K$ is the only evolutionary stable portfolio rule.

Moreover, if $\mathcal{S}$ is the power set of the set of states S, then we find that all other completely mixed adapted strategies are not even locally evolutionary stable.

7 Application to mean-variance optimization and CAPM

The article brings up an application of the theory presented in terms of evolutionary stable market rules. It is shown that, due to under-diversification, using the mean-variance theory of portfolio optimization, the resulting portfolios are not evolutionary stable.

However, using the CAPM-rule (i.e. investing in the market portfolio), will always yield a portfolio with strong resistance against the market selection mechanism, given that all other players use simple strategies (non-random in the unit simplex). This rule can be written as

$$\lambda_{k,t}^{i} = q_{k,t}.$$

This is summarized in a proposition:

Proposition 3. The market share of a CAPM investor is constant in any population in which all other players pursue simple strategies. In particular, a CAPM investor will never vanish nor dominate the market.

The intuition behind the success of the CAPM rule is given in the article, and a simplification of it is as follows: Since asset prices reflect strategies of an asymptotically dominant player (which follows since prices are determined by supply and demand), the CAPM investor mimics the dominant strategy since he invests accordingly to the market value of the assets.

References

[1] 2003, T. Hens and K.R. Schenk-Hoppé. "Evolutionary Stability of Portfolio Rules in Incomplete Markets".