SOLUTIONS TO SELECTED EXERCISES

18th September 2002

Exercise 4.5 A is independent of itself so $\mathbb{P}(A \cap A) = \mathbb{P}(A)^2$. Since $A = A \cap A$ we also have $\mathbb{P}(A) = \mathbb{P}(A)^2$ which can be true only if $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$.

Exercise 4.15 Let $\mathcal{P} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$, i.e. all open intervals of $\mathbb{R}$ and $\mathcal{Q} = \{(a, b] : a, b \in \mathbb{R}, a < b\}$, i.e. all half-open intervals of $\mathbb{R}$. The Borel σ-field is defined as $\mathcal{F} = \sigma(\mathcal{P})$, the σ-field generated by sets in $\mathcal{P}$. Let us denote by $\mathcal{G} = \sigma(\mathcal{Q})$. We want to show that $\mathcal{F} = \mathcal{G}$. First we show $\mathcal{F} \subseteq \mathcal{G}$, i.e. any set in $\mathcal{F}$ can be generated by sets in $\mathcal{G}$. To see this note that for arbitrary $a < b$, there is an integer N such that $b - 1/N > a$ and $(a, b) = \bigcup_{n=N}^{\infty} (a, b - 1/n]$. Next we need to show the opposite inclusion, $\mathcal{G} \subseteq \mathcal{F}$. Indeed, $(a, b] = (a, b) \cup \bigcap_{n=1}^{\infty} (b - 1/n, b + 1/n)$ so any set in $\mathcal{G}$ can be generated by sets in $\mathcal{F}$. Hence, $\mathcal{F} = \mathcal{G}$.

Exercise 7.4 (Correction) The last part in the solution of exercise 7.4 was incorrect. We need to show that $\mathbb{E}(|S_n|) < \infty$ for any n. Since $S_n = \exp(\alpha Y_n - \alpha^2 n/2)$ we have $S_n > 0$ and $E(|S_n|) = \mathbb{E}(S_n)$. Now, $Y_n \sim N(0, n)$ so $\alpha Y_n \sim N(0, \alpha^2 n)$. We can compute $\mathbb{E}(e^{\alpha Y_n}) = e^{\alpha^2 n/2}$. Hence $\mathbb{E}(S_n) = 1 < \infty$.

Exercise 7.7 First note that $X_n \in [0, 1]$ so X_n is bounded and hence $\mathbb{E}(|X_n|) < \infty$. Next the martingale property:

$$\begin{aligned}\mathbb{E}(X_{n+1} | X_0, \ldots, X_n) &= \mathbb{E}(X_{n+1} | X_n) = (p + qX_n)X_n + qX_n(1 - X_n) \\ &= (p + q)X_n = X_n.\end{aligned}$$