SOLUTIONS TO SELECTED EXERCISES

27th August 2002

1 Chapters 1-4

Exercise 4.1 We begin by showing that

$$A, B \in \mathcal{F} \text{ implies } A \cap B \in \mathcal{F}. \tag{1.1}$$

We use the properties (i), (ii) and (iii) in Definition 1.3 p.2. Since $A \cap B = (A^c \cup B^c)^c$ (draw a picture) (1.1) follows as the next argument shows. By (iii) $A^c \in \mathcal{F}$ and $B^c \in \mathcal{F}$. Now, (ii) implies that $A^c \cup B^c \in \mathcal{F}$ and (iii) again completes the argument. Let us show that $B \setminus A \in \mathcal{F}$. Since, $B \setminus A = B \cap A^c$ this follows from (iii) and (1.1). To show that $\cap_{i=1}^{\infty} A_i \in \mathcal{F}$ we note that $\cap_{i=1}^{\infty} A_i = (\cup_{i=1}^{\infty} A_i^c)^c$. Now (iii) implies that $A_i^c \in F$ for each i, (ii) then implies $\cup_{i=1}^{\infty} A_i^c \in \mathcal{F}$ and finally (iii) yields the result.

Exercise 4.6 We have the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$ and want to find the σ-field generated by the events $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. By taking unions, intersections and complements we find that the σ-field is given by

$$\begin{aligned} \mathcal{F} = \big\{&\emptyset, \{2\}, \{5\}, \{1,3\}, \{2,5\}, \{4,6\}, \{1,2,3\}, \{1,3,5\}, \{2,4,6\}, \{4,5,6\}, \\ &\{1,3,4,6\}, \{2,4,5,6\}, \{1,2,3,5\}, \{1,2,3,4,6\}, \{1,3,4,5,6\}, \Omega\big\}. \end{aligned}$$

Exercise 4.12 We have the sample space $\Omega = \{(i, j);\ i = 1, \dots, 6,\ j = 1, \dots, 6\}$. Here i denotes the result of the first throw and j the result of the second throw. We have the given information

$$\begin{aligned} A &= \text{“Two sixes”} = \{(6,6)\} \\ B &= \text{“Exactly one six”} = \{(i,6), (6,j);\ i = 1, \dots, 5,\ j = 1, \dots, 5\} \\ C &= \text{“No sixes”} = \{(i,j);\ i = 1, \dots, 5,\ j = 1, \dots, 5\} \end{aligned}$$

The σ-field, $\mathcal{F}$ generated by A, B and C is given by $\mathcal{F} = \{\emptyset, A, B, C, A^c, B^c, C^c, \Omega\}$ (note that A, B and C are disjoint, $A \cup B = C^c$, $A \cup C = B^c$ and $B \cup C = A^c$). The probabilities of these events are

$$\begin{aligned} &\mathbb{P}(A) = (1/6)(1/6) = 1/36, \quad \mathbb{P}(A^c) = 1 - \mathbb{P}(A) = 35/36 \\ &\mathbb{P}(B) = 2(1/6)(5/6) = 10/36, \quad \mathbb{P}(B^c) = 1 - \mathbb{P}(B) = 26/3611/36 \\ &\mathbb{P}(C) = (5/6)(5/6) = 25/36, \quad \mathbb{P}(C^c) = 1 - \mathbb{P}(C) = 11/36. \end{aligned}$$

Next we determine the $\mathcal{F}$-measurable functions $X : \Omega \to \{-1, 1, 2, \dots\}$ with $\mathbb{E}[X] = 0$. For X the be $\mathcal{F}$-measurable we require that for each $k \in \{-1, 1, 2, \dots\}$, $X^{-1}(k) = \{\omega \mid X(\omega) = k\}$ belongs to $\mathcal{F}$. Since the events A, B and C are disjoint generate the σ-field and $A \cup B \cup C = \Omega$, we only need to specify X on A, B and C. Assume $X(\omega) = k_A$ for $\omega \in A$, $X(\omega) = k_B$ for $\omega \in B$ and $X(\omega) = k_C$ for $\omega \in C$. Since $\mathbb{E}[X] = 0$ we must have

$$0 = \mathbb{E}[X] = k_A\, \mathbb{P}(A) + k_B\, \mathbb{P}(B) + k_C\, \mathbb{P}(C) = k_A \frac{1}{36} + k_B \frac{10}{36} + k_C \frac{25}{36}.$$

For this to hold it is neccessary that $k_C = -1$. Now there are three choices for k_A and k_B. Either, $k_A = 5, k_B = 2$ or $k_A = 15, k_B = 1$ or $k_A = 35, k_B = -1$. This corresponds to the fair games:

$$\begin{aligned} \text{"No sixes"} &\quad\Rightarrow\quad \text{You loose 1 unit} \\ \text{"Exactly one six"} &\quad\Rightarrow\quad \text{You win 2 units} \\ \text{"Two sixes"} &\quad\Rightarrow\quad \text{You win 5 units} \end{aligned}$$

or

$$\begin{aligned} \text{"No sixes"} &\quad\Rightarrow\quad \text{You loose 1 unit} \\ \text{"Exactly one six"} &\quad\Rightarrow\quad \text{You win 1 unit} \\ \text{"Two sixes"} &\quad\Rightarrow\quad \text{You win 15 units} \end{aligned}$$

or

$$\begin{aligned} \text{"No sixes"} &\quad\Rightarrow\quad \text{You loose 1 unit} \\ \text{"Exactly one six"} &\quad\Rightarrow\quad \text{You loose 1 unit} \\ \text{"Two sixes"} &\quad\Rightarrow\quad \text{You win 35 units} \end{aligned}$$

Exercise 4.14 Let Ω denote the sample space and A be a subset of Ω. Let $\mathcal{G}_A$ be the collection of σ-fields $\mathcal{G}$ that contains A. If we show that

$$\mathcal{F} = \cap_{\mathcal{G}\in\mathcal{G}_A}\mathcal{G}$$

is a σ-field, then this is the smallest σ-field that contains the set A. We will now show the slightly more general statement that if Λ is an index set and $\mathcal{F}_\lambda$ is a σ-field for each $\lambda \in \Lambda$, then $\mathcal{F} = \cap_{\lambda\in\Lambda}\mathcal{F}_\lambda$ is a σ-field. Let us verify the conditions (i), (ii) and (iii), in Definition 1.3 p.2. $\emptyset \in \mathcal{F}$ since $\emptyset \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$. This shows (i). Next, if $A_1, A_2, \dots \in \mathcal{F}$ then by property (ii) of a σ-field $\cup_{i=1}^{\infty} A_i \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$ which implies that $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$. This proves (ii). Finally if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}_\lambda$ for each $\lambda \in \Lambda$ and hence $A^c \in \mathcal{F}$.
A counterexample where $\mathcal{F}_1$ and $\mathcal{F}_2$ are σ-fields but $\mathcal{F}_1 \cup \mathcal{F}_2$ is not a σ-field can be constructed as follows. Let $\Omega = \{1, 2, 3\}$ and let $\mathcal{F}_1$ be the σ-field generated by $A = \{1\}$ and $\mathcal{F}_2$ generated by $B = \{2\}$. Clearly, $\mathcal{F}_1 = \{\emptyset, A, A^c, \Omega\}$ and $\mathcal{F}_2 = \{\emptyset, B, B^c, \Omega\}$. Then $A \in \mathcal{F}_1$, $B \in \mathcal{F}_2$ but $A \cup B = \{1, 2\} \notin \mathcal{F}_1$, $A \cup B \notin \mathcal{F}_2$ so $A \cup B \notin \mathcal{F}_1 \cup \mathcal{F}_2$ which contradicts the property (ii) of a σ-field.

Exercise 4.21 We have $\hat{X} = \mathbb{E}[X \mid \mathcal{A}]$ and Y is $\mathcal{A}$-measurable. Using the properties (ii) and (iii) in Proposition 4.8 p.21-22 we find,

$$\begin{aligned} \mathbb{E}[(X - \hat{X})Y] &= \mathbb{E}\big[(X - E[X \mid \mathcal{A}])Y\big] = \mathbb{E}\big[XY - E[XY \mid \mathcal{A}]\big] \\ &= \mathbb{E}[XY] - \mathbb{E}\big[\mathbb{E}[XY \mid \mathcal{A}]\big] = \mathbb{E}[XY] - \mathbb{E}[XY] = 0, \end{aligned}$$

which proves (i). To prove (ii) let $Y = \hat{X} + Z$ for some $\mathcal{A}$-measurable random variable Z. If we can prove that $Z = 0$ almost surely, then the statement follows. We have the squared error,

$$\begin{aligned} \mathbb{E}\big[(X - Y)^2\big] &= \mathbb{E}\big[(X - \hat{X} - Z)^2\big] = \mathbb{E}\big[(X - \hat{X})^2\big] - 2\mathbb{E}\big[(X - \hat{X})Z\big] + \mathbb{E}[Z^2] \\ &= \mathbb{E}\big[(X - \hat{X})^2\big] + \mathbb{E}[Z^2], \end{aligned}$$

since $\mathbb{E}\big[(X - \hat{X})Z\big] = 0$ by (i). We note that this expression is minimized when $\mathbb{E}[Z^2] = 0$, that is $Z = 0$ almost surely.

Exercise 4.22 (i) Let $\mathcal{F}^X$ be the σ-field generated by X. We sometimes write $\mathbb{E}[Y \mid X]$ for $\mathbb{E}[Y \mid \mathcal{F}^X]$. We will use property (iv) of Proposition 4.8 pp. 21-22. Since X is assumed to be $\mathcal{A}$-measurable we have $\mathcal{F}^X \subseteq \mathcal{A}$. Suppose now that $\mathbb{E}[Y \mid \mathcal{A}] = X$. Then,

$$\mathbb{E}[Y \mid X] = \mathbb{E}[Y \mid \mathcal{F}^X] = \{\text{(iv)}\} = \mathbb{E}\big[\mathbb{E}[Y \mid \mathcal{A}] \,\big|\, \mathcal{F}^X\big] = \mathbb{E}[X \mid \mathcal{F}^X] = X.$$

(ii) We can construct a counterexample as follows. Consider the 'throw of fair die'. That is, we have the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\mathbb{P}(\omega = k) = 1/6$ for each $k = 1, \dots, 6$. Let $A = \{1, 2, 3\}$ and X be the random variable

$$X(\omega) = \begin{cases} 2 & \text{if } \omega \in A, \\ 5 & \text{if } \omega \in A^c. \end{cases}$$

Let Y be the random variable $Y(\omega) = \omega$ and $\mathcal{A}$ be the σ-field consisting of all subsets of Ω. Clearly, X is $\mathcal{A}$-measurable. Furthermore, $\mathbb{E}[Y \mid \mathcal{F}^X] = X$ since

$$\mathbb{E}[Y \mid \mathcal{F}^X](\omega) = \mathbb{E}[Y \mid A]\mathbf{1}_A(\omega) + \mathbb{E}[Y \mid A^c]\mathbf{1}_{A^c}(\omega) = \begin{cases} 2 & \text{if } \omega \in A, \\ 5 & \text{if } \omega \in A^c. \end{cases}$$

But $\mathbb{E}[Y \mid \mathcal{A}](\omega) = Y(\omega) \neq X(\omega)$ for $\omega \in \{1, 3, 4, 6\}$. This completes the counter example.

2 Chapters 5-7

Exercise 7.3 Since it is not specified in the exercise we have to assume that $\mathbb{E}[|Y_n|] < \infty$ for each $n \geq 1$. Next we check the martingale property in the cases (i)-(iv).

$$\begin{aligned} (i) \quad & \mathbb{E}[Y_{n+1} \mid X_1, \dots, X_n] = \mathbb{E}[X_{n+1} + Y_n \mid X_1, \dots, X_n] \\ & = \mathbb{E}[X_{n+1} \mid X_1, \dots, X_n] + \mathbb{E}[Y_n \mid X_1, \dots, X_n] = 0 + Y_n = Y_n. \end{aligned}$$

Hence, (i) is a martingale.

$$\begin{aligned} (ii) \quad & \mathbb{E}[Y_{n+1} \mid X_1, \dots, X_n] = \mathbb{E}[\frac{1}{n+1}\sum_{i=1}^{n+1} X_i \mid X_1, \dots, X_n] \\ & = \mathbb{E}[\frac{1}{n+1}(X_{n+1} + nY_n) \mid X_1, \dots, X_n] \\ & = \frac{1}{n+1}\mathbb{E}[X_{n+1} \mid X_1, \dots, X_n] + \frac{n}{n+1}\mathbb{E}[Y_n \mid X_1, \dots, X_n] \\ & = 0 + \frac{n}{n+1}Y_n = \frac{n}{n+1}Y_n \neq Y_n. \end{aligned}$$

Hence, (ii) is *not* a martingale.

$$\begin{aligned} (iii) \quad & \mathbb{E}[Y_{n+1} \mid X_1, \dots, X_n] = \mathbb{E}[X_{n+1}Y_n \mid X_1, \dots, X_n] \\ & = \mathbb{E}[X_{n+1}]\mathbb{E}[Y_n \mid X_1, \dots, X_n] = 0 \neq Y_n, \end{aligned}$$

since $\mathbb{E}[X_{n+1}] = 0$. Hence, Y_n is *not* a martingale. However, if we have $\mathbb{E}[X_i] = 1$ for each $i \geq 1$, then Y_n becomes a martingale.

(iv) If we put $Z_i = \exp(X_i)$, then this is the same situation as in (iii) with $Y_n = \prod_{i=1}^n Z_i$. Hence, it is a martingale if $\mathbb{E}[Z_i] = 1$. We need to know the distribution of X_i to be able to verify if $\mathbb{E}[Z_i] = 1$ or not.

Exercise 7.4 We begin to verify the martingale property.

$$
\begin{aligned}
\mathbb{E}[S_n \mid X_1, \ldots, X_n] &= \mathbb{E}[\exp\{\alpha \sum_{i=1}^{n+1} X_i - \frac{(n+1)\alpha^2}{2} \mid X_1, \ldots, X_n] \\
&= \mathbb{E}[\exp\{\alpha \sum_{i=1}^{n} X_i - \frac{n\alpha^2}{2} \mid X_1, \ldots, X_n] \times \mathbb{E}[\exp\{\alpha X_{n+1} - \frac{\alpha^2}{2}] \\
&= S_n \mathbb{E}[\exp\{\alpha X_{n+1} - \frac{\alpha^2}{2}\}].
\end{aligned}
$$

It has the martingale property if $\mathbb{E}[\exp\{\alpha X_{n+1} - \frac{\alpha}{2}\}] = 1$. Since each $X_i \sim N(0,1)$ we have,

$$
\begin{aligned}
\mathbb{E}[\exp\{\alpha X_{n+1} - \frac{\alpha^2}{2}\}] &= e^{-\alpha^2/2} \int_{-\infty}^{\infty} e^{\alpha x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= e^{-\alpha^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha)^2 + \alpha^2/2} dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\alpha)^2} dx = 1.
\end{aligned}
$$

Hence, the martingale property is verified. Finally we need to show that $\mathbb{E}[|S_n|] < \infty$ for each $n \geq 1$. Since $S_n \geq 0$ for each n we have $\mathbb{E}[|S_n|] = \mathbb{E}[S_n]$. The martingale property yields

$$
\mathbb{E}[S_n] = \mathbb{E}\big[\mathbb{E}[S_n \mid X_1]\big] = \mathbb{E}[S_1] = 1 < \infty.
$$

Hence, S_n is a martingale.

Exercise 7.5 Since we have assumed that $\mathbb{E}[|\varphi(S_n)|] < \infty$ holds we only need to verify the sub-martingale property, i.e. that $\mathbb{E}[\varphi(S_{n+1}) \mid X_1, \ldots, X_n] \geq \varphi(S_n)$. By Jensen's inequality, Theorem 4.10 p. 23, and the martingale property for S_n we have

$$
\mathbb{E}[\varphi(S_{n+1}) \mid X_1, \ldots, X_n] \geq \varphi(\mathbb{E}[S_{n+1} \mid X_1, \ldots, X_n]) = \varphi(S_n).
$$

Hence, $\varphi(S_n)$ is a sub-martingale.

3 Chapter 9

Exercise 9.3 First we notice that $S_n = s \cdot \prod_{i=1}^{n}(1+R_i)$ and that $B_n = (1+r)^n$.
(a) Let us verify the martingale property for the sequence S_n/B_n.

$$
\begin{aligned}
\mathbb{E}[\frac{S_{n+1}}{B_{n+1}} \mid \mathcal{F}_n] &= \mathbb{E}[\frac{s\prod_{i=1}^{n+1}(1+R_i)}{(1+r)^{n+1}} \mid \mathcal{F}_n] \\
&= \mathbb{E}[\frac{s\prod_{i=1}^{n}(1+R_i)}{(1+r)^{n}} \cdot \frac{1+R_{n+1}}{1+r} \mid \mathcal{F}_n] \\
&= \frac{S_n}{B_n}\mathbb{E}[\frac{1+R_{n+1}}{1+r} \mid \mathcal{F}_n] = \frac{S_n}{B_n},
\end{aligned}
$$

if $\mathbb{E}[R_n] = r$. Hence, S_n/B_n has the martingale property if $\mathbb{E}[R_n] = r$. (We have to assume that the R_n's are such that $\mathbb{E}[|S_n/B_n|] < \infty$ to make sure it is a martingale).

(b) Assume $\mathbb{E}[R_n] = r$. We have,

$$\begin{aligned}
\mathbb{E}[\frac{V_{n+1}}{B_{n+1}} \mid \mathcal{F}_n] &= \mathbb{E}[\frac{x_{n+1}S_{n+1} + y_{n+1}B_{n+1}}{B_{n+1}} \mid \mathcal{F}_n] = \{\text{self-financing condition}\} \\
&= \mathbb{E}[\frac{x_n S_{n+1} + y_n B_{n+1}}{B_{n+1}} \mid \mathcal{F}_n] = x_n \mathbb{E}[\frac{S_{n+1}}{B_{n+1}} \mid \mathcal{F}_n] + y_n \\
&= x_n \frac{S_n}{B_n} + y_n = \frac{x_n S_n + y_n B_n}{B_n} = \frac{V_n}{B_n}.
\end{aligned}$$

Hence, V_n/B_n is a martingale.
(c) Since V_n/B_n is a martingale it follows that $\mathbb{E}[V_N/B_N] = \mathbb{E}[V_0/B_0] = 0$. Hence, $\mathbb{E}[V_N] = 0$. But if $\mathbb{P}(V_N \geq 0) = 1$ and $\mathbb{P}(V_n > 0) > 0$, then we would have $\mathbb{E}[V_N] > 0$ which is impossible since $\mathbb{E}[V_N] = 0$. Thus, there can be not arbitrage-stretegies in the market.

Exercise 9.4 (a) We get

$$\begin{aligned}
\mathbb{E}[S_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[S_n + C_n X_{n+1} \mid \mathcal{F}_n] = S_n + C_n \mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \\
&= S_n + C_n(p \cdot 1 + (1-p)\cdot(-1)) = S_n + C_n(2p-1) \geq S_n.
\end{aligned}$$

(b) Now,

$$\begin{aligned}
\mathbb{E}[L_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[\log S_{n+1} - \alpha(n+1) \mid \mathcal{F}_n] \\
&= \mathbb{E}[\log(S_{n+1} + C_n X_{n+1}) \mid \mathcal{F}_n] - \alpha(n+1) \\
&= \mathbb{E}[\log S_{n+1} + \log\big(1 + \frac{C_n}{S_n} X_{n+1}\big) \mid \mathcal{F}_n] - \alpha(n+1) \\
&= \log S_n - \alpha n + \mathbb{E}[\log\big(1 + \frac{C_n}{S_n} X_{n+1}\big) \mid \mathcal{F}_n] - \alpha \\
&= L_n + p\log\big(1 + \frac{C_n}{S_n}\big) + (1-p)\log\big(1 - \frac{C_n}{S_n}\big) - \alpha.
\end{aligned}$$

Since $0 \leq C_n \leq S_n \iff 0 \leq C_n/S_n \leq 1$ wee see that if we can show that

$$g(x) = p\log(1+x) + (1-p)\log(1-x) \leq \alpha$$

for $x \in [0,1]$ and $p \in (\frac{1}{2}, 1)$ then we are done. Now $g(0) = 0$ and

$$\begin{aligned}
g'(x) &= \frac{p}{1+x} - \frac{1-p}{1-x} = \frac{2p-1-x}{1-x^2} \\
g''(x) &= -\frac{x^2 + 2(2p-1)x + 1}{(1-x^2)^2} < 0, \quad x \in [0,1].
\end{aligned}$$

Hence, the function g is concave with maximum at $\hat{x} = 2p-1$ and since

$$\begin{aligned}
g(x) \leq g(\hat{x}) &= p\log(1+2p-1) + (1-p)\log(1-2p+1) \\
&= p\log 2 + p\log p + (1-p)\log 2 + (1-p)\log(1-p) \\
= \alpha
\end{aligned}$$

we are done.
(c) We know that $L_n = \log S_n - \alpha n$ is a supermartingale;

$$\mathbb{E}[L_N] \leq \mathbb{E}[L_0] \iff \mathbb{E}[\log S_N - \alpha N] \leq \mathbb{E}[\log S_0 - \alpha \cdot 0] = \log S_0,$$

and from this we get

$$\mathbb{E}[\log S_N - \log S_0] \leq \alpha N \iff \mathbb{E}[\log(S_N/S_0)] \leq \alpha N.$$

(d) Now we shall use the explicit strategy $C_n = S_n(2p-1)$. The chosen strategy implies

$$S_{n+1} = S_n + C_n \cdot X_{n+1} = S_n[1 + (2p-1)X_{n+1}],$$

and we get

$$\begin{aligned}
\mathbb{E}[\log S_{n+1} - \alpha(n+1) \mid \mathcal{F}_n] &= \mathbb{E}[\log S_n + \log(1 + (2p-1)X_{n+1}) \mid \mathcal{F}_n] - \alpha(n+1) \\
&= \log S_n + p \cdot \log(1 + (2p-1) \cdot 1) \\
&\quad +(1-p) \cdot \log(1 + (2p-1) \cdot (-1)) - \alpha(n+1) \\
&= \log S_n - \alpha n + p \log(2p) + (1-p) \log(2(1-p)) - \alpha \\
&= \log S_n - \alpha n.
\end{aligned}$$

Exercise 9.5 We will use the super-martingale property for X_n, i.e. that $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \leq X_n$ or equivalently that $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - X_n \leq 0$. We have,

$$\begin{aligned}
\mathbb{E}[I_X(C)_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[\sum_{k=0}^{n} C_k(X_{k+1} - X_k) \mid \mathcal{F}_n] \\
&= \mathbb{E}[C_n(X_{n+1} - X_n) + I_X(C)_n \mid \mathcal{F}_n] \\
&= C_n(\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] - X_n) + I_X(C)_n \\
&\leq I_X(C)_n.
\end{aligned}$$

4 Chapter 10

Exercise 10.1 We will use Markov's inequality (see e.g. A. Gut, An intermediate course in probability p. 12); for a random variable X with mean m and variance σ^2 one has for all $\varepsilon > 0$ that

$$\mathbb{P}(|X - m| > \varepsilon) \leq \sigma^2/\varepsilon^2.$$

Now, the discrete Brownian motion has mean 0 and variance $\mathbb{E}[B_n^2] = n$. Furthermore, it has quadratic variation $\langle B \rangle_n = n$. Hence,

$$\mathbb{P}\left\{\left|\frac{B_n}{\langle B \rangle_n}\right| > \varepsilon\right\} = \mathbb{P}\{|B_n| > \varepsilon n\} \leq \frac{n}{\varepsilon^2 n^2} = \frac{1}{\varepsilon^2 n} \to 0, \quad \text{as } n \to \infty.$$

Exercise 10.2 First note that $\tilde{S}_n = S_n/B_n$ (this is not well specified in the formulation). We write the discrete Brownian motion as $W_n = \sum_{i=1}^{n} X_i$ where the X_i's are i.i.d. $N(0,1)$ random variables. If we put $B_n = (1+r)^n$ with $r = \exp(\sigma^2/2) - 1$, then $B_n = \exp(n\sigma^2/2)$. Hence,

$$\begin{aligned}
\tilde{S}_n &= e^{\sigma \sum_{i=1}^{n} X_i - \frac{1}{2} n \sigma^2} = \frac{e^{\sigma \sum_{i=1}^{n} X_i}}{B_n} = \frac{\prod_{i=1}^{n} e^{\sigma X_i}}{B_n} \\
&= \frac{\prod_{i=1}^{n}(1 + R_i)}{B_n} = \frac{S_n}{B_n},
\end{aligned}$$

with $1 + R_i = e^{\sigma X_i}$. So the random variable $1 + R_i$ has lognormal distribution with parameters 0 and σ^2. This is usually written as $1 + R_i \sim \textit{lognormal}(0, \sigma^2)$.

5 Chapter 11

Exercise 11.1 (i) We will use Theorem 11.10 on p. 75 which says that $\langle M\rangle_t$ is the unique continuous increasing adapted process such that $M_t^2 - \langle M\rangle_t$ is an $\{\mathcal{F}_t\}$-martingale. This means that $\langle \alpha M + \beta N\rangle_t$ is the unique continuous increasing adapted process such that $(\alpha M_t + \beta N_t)^2 - \langle \alpha M + \beta N\rangle_t$ is a martingale. But we also have that

$$(\alpha M_t + \beta N_t)^2 = \alpha^2 M_t^2 + 2\alpha\beta M_t N_t + \beta^2 N_t^2.$$

It follows that the process $\alpha^2\langle M\rangle_t + 2\alpha\beta\langle M, N\rangle_t + \beta^2\langle N\rangle_t$ is the unique continuous increasing adapted process such that $(\alpha M_t + \beta N_t)^2 - (\alpha^2\langle M\rangle_t + 2\alpha\beta\langle M, N\rangle_t + \beta^2\langle N\rangle_t)$ is a martingale. Hence, by uniqeness $\langle \alpha M + \beta N\rangle_t$ and $\alpha^2\langle M\rangle_t + 2\alpha\beta\langle M, N\rangle_t + \beta^2\langle N\rangle_t$ must coincide.

(ii) For any sequence of real numbers $\{a_j\}$ and $\{b_j\}$ we have the Cauchy-Schwartz inequality

$$\Big|\sum_j a_j b_j\Big| \le \sum_j |a_j b_j| \le \Big(\sum_j a_j^2\Big)^{1/2}\Big(\sum_j b_j^2\Big)^{1/2}.$$

By using the definition of the quadratic covariation we get that

$$\begin{aligned}
|\langle M, N\rangle_t| &\overset{\mathbb{P}}{=} \lim_{\|\Pi\|\to 0} \Big|\sum_{k=0}^{n-1}(M_{t_{j+1}} - M_{t_j})(N_{t_{j+1}} - N_{t_j})\Big| \\
&\le \lim_{\|\Pi\|\to 0} \sum_{k=0}^{n-1} |(M_{t_{j+1}} - M_{t_j})(N_{t_{j+1}} - N_{t_j})| \\
&\le \lim_{\|\Pi\|\to 0} \Big(\sum_{k=0}^{n-1}(M_{t_{j+1}} - M_{t_j})^2\Big)^{1/2} \lim_{\|\Pi\|\to 0} \Big(\sum_{k=0}^{n-1}(N_{t_{j+1}} - N_{t_j})^2\Big)^{1/2} \\
&= \sqrt{\langle M\rangle_t}\sqrt{\langle N\rangle_t}.
\end{aligned}$$

6 Chapter 12

Exercise 12.1 First we remark that the function $(S_T - K)^+ = \max(S_T - K; 0)$. Let us denote the density of a $N(0,1)$ random variable by $\varphi(z)$ and the distribution function by $\Phi(z)$. Note that

$$S_T = \exp\{(r - \sigma^2/2)T + \sigma B_T\} = S_t \exp\{(r - \sigma^2/2)(T-t) + \sigma(B_T - B_t)\}.$$

Now we have,

$$\begin{aligned}
\mathbb{E}[\max(S_T - K; 0) \mid \mathcal{F}_t] &= \mathbb{E}[\max(S_t e^{\{(r-\sigma^2/2)(T-t)+\sigma(B_T - B_t)\}} - K; 0) \mid \mathcal{F}_t] \\
&= \int_{z=-\infty}^{\infty} \max(S_t e^{\{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\cdot z\}} - K; 0)\varphi(z)dz \\
&= \int_{z=d_2}^{\infty} (S_t e^{\{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\cdot z\}} - K)\varphi(z)dz,
\end{aligned}$$

where d_2 is the solution to $S_t e^{\{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\cdot d_2\}} - K = 0$. That is,

$$d_2 = \frac{\ln(K/S_t) - (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Continuing the computation above we see that the last expression equals

$$
\begin{aligned}
&\int_{z=d_2}^{\infty} S_t e^{\{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\cdot z\}}\varphi(z)dz - \int_{z=d_2}^{\infty} K\varphi(z)dz \\
&\qquad = S_t e^{r(T-t)} \int_{z=d_2}^{\infty} e^{-\sigma^2(T-t)/2+\sigma\sqrt{T-t}\cdot z}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz - K\int_{z=d_2}^{\infty}\varphi(z)dz \\
&\qquad = S_t e^{r(T-t)} \int_{z=d_2}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z-\sigma\sqrt{T-t})^2}dz - K\int_{z=d_2}^{\infty}\varphi(z)dz \\
&\qquad = S_t e^{r(T-t)} \int_{u=d_2-\sigma\sqrt{T-t}}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-u^2/2}du - K\int_{z=d_2}^{\infty}\varphi(z)dz \\
&\qquad = S_t e^{r(T-t)}(1-\Phi(d_2-\sigma\sqrt{T-t})) - K(1-\Phi(d_2)) \\
&\qquad = S_t e^{r(T-t)}\Phi(-d_2+\sigma\sqrt{T-t})) - K\Phi(-d_2).
\end{aligned}
$$

If we put $d_1 = d_2 - \sigma\sqrt{T-t}$ we get the expression

$$
\mathbb{E}[\max(S_T - K; 0) \mid \mathcal{F}_t] = S_t e^{r(T-t)}\Phi(-d_1) - K\Phi(-d_2).
$$

Exercise 12.2 Let us first show that $\mathbb{E}[B_t^2 \mid \mathcal{F}_s] = B_s^2 + t - s$. Let $0 \le s < t$.

$$
\begin{aligned}
\mathbb{E}[B_t^2 \mid \mathcal{F}_s] &= \mathbb{E}[(B_t - B_s)^2 + 2B_tB_s - B_s^2 \mid \mathcal{F}_s] = t - s + 2B_s\mathbb{E}[B_t \mid \mathcal{F}_s] - B_s^2 \\
&= B_s^2 + t - s.
\end{aligned}
$$

Let $0 \le s < t$ and use the result above to get

$$
\begin{aligned}
\mathbb{E}[B_t^3 \mid \mathcal{F}_s] &= \mathbb{E}[(B_t - B_s)^3 - 3B_tB_s^2 + 3B_t^2B_s + B_s^3 \mid \mathcal{F}_s] \\
&= 0 - 3B_s^2\mathbb{E}[B_t \mid \mathcal{F}_s] + 3B_s\mathbb{E}[B_t^2 \mid \mathcal{F}_s] + B_s^3 \\
&= -3B_s^3 + 3B_s(B_s^2 + t - s) + B_s^3 \\
&= B_s^3 + 3(t-s)B_s.
\end{aligned}
$$

Exercise 12.3 We will use the formulas from Exercise 12.2.
(a) Let $0 \le s < t$.

$$
\begin{aligned}
\mathbb{E}[B_t^3 - 3tB_t \mid \mathcal{F}_s] &= \mathbb{E}[B_t^3 \mid \mathcal{F}_s] - 3t\mathbb{E}[B_t\mathcal{F}_s] \\
&= B_s^3 + 3(t-s)B_s - 3tB_s \\
&= B_s^3 - 3sB_s.
\end{aligned}
$$

(b) Let $0 \le s < t$.

$$
\begin{aligned}
\mathbb{E}[B_t^4 - 6tB_t^2 + 3t^2 \mid \mathcal{F}_s] &= \mathbb{E}[(B_t - B_s)^4 + 4B_t^3B_s - 6B_t^2B_s^2 + 4B_tB_s^3 - B_s^4 - 6tB_t^2 + 3t^2 \mid \mathcal{F}_s] \\
&= 3(t-s)^2 + 4B_S\mathbb{E}[B_t^3 \mid \mathcal{F}_s] - 6(B_s^2 + t)\mathbb{E}[B_t^2 \mid \mathcal{F}_s] + 4B_s^4 - B_s^4 + 3t^2 \\
&= 3(t-s)^2 + 4B_s(B_s^3 + 3(t-s)B_s) - 6(B_s^2 + t)(B_s^2 + t - s) + 3B_s^4 + 3t^2 \\
&= 3(t-s)^2 + 4B_s^4 + 12B_s^2(t-s) - 6B_s^4 - 6B_s^2(t-s) - 6tB_s^2 - 6t(t-s) + 3B_s^4 + 3t^2 \\
&= 3t^2 - 6ts + 3s^2 + B_s^4 + 6tB_s^2t - 6sB_s^2 - 6tB_s^2 - 6t^2 + 6ts + 3t^2 \\
&= B_s^4 - 6sB_s^2 + 3s^2.
\end{aligned}
$$

7 Chapter 13

Exercise 13.2 There is a misprint in the formulation. It should be

$$
X_t = \int_0^t e^{s-t}dB_s.
$$

(a) We will use Proposition 13.12 on p. 97 which says that if f is deterministic, then $I_t(f) \sim N(0, \int_0^t f^2(s)ds)$. Let us put $f(s) = e^s$. Then we have,

$$X_t = \int_0^t e^{s-t} dB_s = e^{-t} \int_0^t f(s) dB_s.$$

Hence, X_t has normal distribution with mean 0 and variance

$$Var(X_t) = Var(e^{-t} I_t(f)) = e^{-2t} \int_0^t f^2(s)ds = e^{-2t} \int_0^t e^{2s} ds = \frac{1 - e^{-2t}}{2}.$$

(b) Put $W_t = \sqrt{2(t+1)} X_{\left(\frac{1}{2}\log(t+1)\right)}$. Since $X_t \sim N(0, \frac{1-e^{-2t}}{2})$ we see that $X_{\left(\frac{1}{2}\log(t+1)\right)}$ has normal distribution with mean 0 and variance

$$Var(X_{\left(\frac{1}{2}\log(t+1)\right)}) = \frac{1 - e^{-2\frac{1}{2}\log(t+1)}}{2} = \frac{1 - 1/(t+1)}{2}.$$

It follows that $W_t = \sqrt{2(t+1)} X_{\left(\frac{1}{2}\log(t+1)\right)}$ has normal distribution with mean 0 and variance

$$\begin{aligned} Var(W_t) &= Var(\sqrt{2(t+1)} X_{\left(\frac{1}{2}\log(t+1)\right)}) = 2(t+1) Var(X_{\left(\frac{1}{2}\log(t+1)\right)}) \\ &= 2(t+1) \frac{1 - 1/(t+1)}{2} = t. \end{aligned}$$

8 Chapter 15

Exercise 15.2 We will use the Lévy characterization, Corollary 15.6 on p. 117. Clearly X_t is a continuous martingale since it is a sum of Itô integrals. It remains to compute the quadratic variation of X_t. First note that

$$1 = (OO^T)_{ii} = \sum_{k=1}^n O_{ik} O_{ki}^T = \sum_{k=1}^n O_{ik}^2$$

and for $i \neq j$

$$0 = (OO^T)_{ij} = \sum_{k=1}^n O_{ik} O_{kj}^T = \sum_{k=1}^n O_{ik} O_{jk}$$

since $O(t)$ is an orthogonal matrix. Now

$$\langle X^i \rangle_t = \left\langle \sum_{k=1}^n \int_0^\cdot O_{ik}(s) dB_s^k \right\rangle_t = \int_0^t \sum_{k=1}^n O_{ik}^2(s) ds = t$$

and with $i \neq j$

$$\begin{aligned} \langle X^i, X^j \rangle_t &= \left\langle \sum_{k=1}^n \int_0^\cdot O_{ik}(s) dB_s^k, \sum_{l=1}^n \int_0^\cdot O_{jl}(s) dB_s^l \right\rangle_t = \int_0^t \sum_{k=1}^n O_{ik} O_{jk}(s) ds \\ &= 0. \end{aligned}$$

Thus, we see that the n-dimensioinal process X_t has quadratic variation that can be written

$$\langle X \rangle_t = \begin{cases} t & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence, by Corollary 15.6, X_t is an n-dimensional Brownian motion.

Exercise 15.3 (a) We want to find $\varphi(s,\omega)$ and z such that

$$B_T^3(\omega) = z + \int_0^T \varphi(s,\omega)dB(s).$$

Put $g(x) = x^3$. Then

$$\frac{dg}{dx} = 3x^2, \qquad \frac{d^2g}{dx^2} = 6x.$$

Using Itô's formula we get,

$$dg(B_t) = 3B_t^2 dB_t + 3B_t dt.$$

Integrating yields,

$$B_t^3 = 3\int_0^t B_s^2 dB_s + 3\int_0^t B_s ds.$$

It remains to rewrite the integral $\int_0^t B_s ds$ as an Itô integral w.r.t. dB. From Example 14.4 on p. 103 we know that

$$\int_0^t B_s ds = tB_t - \int_0^t s dB_s = \int_0^t (t-s)dB_s.$$

It follows that

$$\begin{aligned}
B_T^3 &= 3\int_0^T B_s^2 dB_s + 3\int_0^T B_s ds \\
&= 3\int_0^T B_s^2 dB_s + 3\int_0^T (T-s)dB_s \\
&= \int_0^T 3(B_s^2 + T - s)dB_s.
\end{aligned}$$

Hence,

$$B_T^3(\omega) = z + \int_0^T \varphi(s,\omega)dB(s),$$

with $z = 0$ and $\varphi(s,\omega) = 3(B_s^2(\omega) + T - s)$.

(b) We want to find $\varphi(s,\omega)$ and z such that

$$\int_0^T B_s^3 ds = z + \int_0^T \varphi(s,\omega)dB(s).$$

Put $g(t,x) = tx^3$. Then

$$\frac{\partial g}{\partial t} = x^3, \qquad \frac{\partial g}{\partial x} = 3tx^2, \qquad \frac{\partial^2 g}{\partial x^2} = 6tx.$$

Using Itô's formula we get,

$$dg(t,B_t) = B_t^3 dt + 3tB_t^2 dB_t + 3tB_t dt.$$

Integrating yields,

$$tB_t^3 = \int_0^t B_s^3 ds + 3\int_0^t sB_s^2 dB_s + 3\int_0^t sB_s ds.$$

Rewriting this equation we find that

$$\int_0^t B_s^3 ds = tB_t^3 - 3\int_0^t sB_s^2 dB_s - 3\int_0^t sB_s ds. \tag{8.1}$$

It remains to rewrite the integral $\int_0^t sB_s ds$ and the term tB_t^3 as Itô integrals w.r.t. dB. We start with the term $\int_0^t sB_s ds$. Let us put $h(t,x) = t^2 x$. Then,

$$\frac{\partial h}{\partial t} = 2tx, \qquad \frac{\partial h}{\partial x} = t^2, \qquad \frac{\partial^2 h}{\partial x^2} = 0.$$

Itô's formula gives,

$$dh(t, B_t) = 2tB_t dt + t^2 dB_t.$$

Integrating yields,

$$t^2 B_t = 2\int_0^t sB_s ds + \int_0^t s^2 dB_s.$$

Hence,

$$\int_0^t sB_s ds = \frac{1}{2}t^2 B_t - \frac{1}{2}\int_0^t s^2 dB_s.$$

We know from (a) that $B_t^3 = \int_0^t 3(B_s^2 + t - s)dB_s$. Substituting in equation (8.1) we get

$$\begin{aligned}
\int_0^t B_s^3 ds &= tB_t^3 - 3\int_0^t sB_s^2 dB_s - 3\int_0^t sB_s ds \\
&= tB_t^3 - 3\int_0^t sB_s^2 dB_s - \frac{3}{2}\Big(t^2 B_t - \int_0^t s^2 dB_s\Big) \\
&= t\int_0^t 3(B_s^2 - s + t)dB_s - 3\int_0^t sB_s^2 dB_s - \frac{3}{2}\Big(t^2 B_t - \int_0^t s^2 dB_s\Big).
\end{aligned}$$

Finally we see that

$$\begin{aligned}
\int_0^T B_s^3 ds &= T\int_0^T 3(B_s^2 - s + T)dB_s - 3\int_0^T sB_s^2 dB_s - \frac{3}{2}\Big(T^2 B_T - \int_0^T s^2 dB_s\Big) \\
&= \int_0^T \{3T(B_s^2 - s + T) - 3sB_s^2 - \frac{3}{2}T^2 + \frac{3}{2}s^2\} dB_s.
\end{aligned}$$

Hence,

$$\int_0^T B_s^3 ds = z + \int_0^T \varphi(s,\omega) dB(s),$$

with $z = 0$ and $\varphi(s,\omega) = 3T(B_s^2 - s + T) - 3sB_s^2 - \frac{3}{2}T^2 + \frac{3}{2}s^2$.

(c) We want to find $\varphi(s,\omega)$ and z such that

$$e^{T/2}\cosh(B_T(\omega)) = z + \int_0^T \varphi(s,\omega) dB(s).$$

Recall that $\cosh(x) = (e^x + e^-x)/2$. If we can find z_1, z_2, $\varphi_1(s,\omega)$ and $\varphi_2(s,\omega)$ such that

$$e^{B_T(\omega)} = z_1 + \int_0^T \varphi_1(s,\omega)dB(s), \qquad e^{-B_T(\omega)} = z_2 + \int_0^T \varphi_2(s,\omega)dB(s).$$

Then we can take $z = e^{T/2}(z_1 + z_2)/2$ and $\varphi(s,\omega) = e^{T/2}(\varphi_1(s,\omega) + \varphi_2(s,\omega))/2$. Let us start by finding $z_!$ and φ_1. Put $g(t,x) = e^{x-\frac{1}{2}t}$. Then,

$$\frac{\partial g}{\partial t} = -\frac{1}{2}e^{x-\frac{1}{2}t}, \qquad \frac{\partial g}{\partial x} = e^{x-\frac{1}{2}t}, \qquad \frac{\partial^2 g}{\partial x^2} = e^{x-\frac{1}{2}t}.$$

Using Itô's formula we get,

$$dg(t,B_t) = -\frac{1}{2}e^{B_t-\frac{1}{2}t}dt + e^{B_t-\frac{1}{2}t}dB_t + \frac{1}{2}e^{B_t-\frac{1}{2}t}dt.$$

Integrating yields,

$$e^{B_t-\frac{1}{2}t} - 1 = \int_0^t e^{B_s-\frac{1}{2}s}dB_s.$$

Hence,

$$e^{B_T} = e^{T/2} + e^{T/2}\int_0^T e^{B_s-\frac{1}{2}s}dB_s.$$

Consequently, $z_1 = e^{T/2}$ and $\varphi_1(s,\omega) = e^{T/2}e^{B_s(\omega)-\frac{1}{2}s}$. For the term e^{-B_T} we take $g(t,x) = e^{-x-\frac{1}{2}t}$. Proceeding analogously we find that

$$e^{-B_T} = e^{T/2} - e^{T/2}\int_0^T e^{-B_s-\frac{1}{2}s}dB_s.$$

Consequently, $z_2 = e^{T/2}$ and $\varphi_2(s,\omega) = -e^{T/2}e^{-B_s(\omega)-\frac{1}{2}s}$. Combining these two expressions we get the representation

$$e^{T/2}\cosh(B_T(\omega)) = z + \int_0^T \varphi(s,\omega)dB(s),$$

with $z = e^T$ and

$$\begin{aligned}\varphi(s,\omega) &= e^{T/2}(\varphi_1(s,\omega) + \varphi_2(s,\omega))/2 = e^{T/2}(e^{T/2}e^{B_s(\omega)-\frac{1}{2}s} - e^{T/2}e^{-B_s(\omega)-\frac{1}{2}s})/2 \\ &= e^T e^{-\frac{1}{2}s}(e^{B_s(\omega)} - e^{-B_s(\omega)})/2 = e^{T-\frac{1}{2}s}\sinh(B_s(\omega)).\end{aligned}$$

Exercise 15.4 (a) We start by defining $M_t = \exp\{B_t^1 + \cdots + B_t^n - nt/2\}$. Then $F(\omega) = M_T(\omega)\exp(nT/2)$ and by Itô's formula

$$\begin{aligned}dM_t =& -\frac{n}{2}\exp\{B_t^1 + \cdots + B_t^n - nt/2\}dt + \sum_{i=1}^n \big(\exp\{B_t^1 + \cdots + B_t^n - nt/2\}dB_t^i \\ &+ \frac{1}{2}\exp\{B_t^1 + \cdots + B_t^n - nt/2\}dt\big) \\ =& -\frac{n}{2}M_t dt + M_t\sum_{i=1}^n dB_t^i + \frac{n}{2}M_t dt \\ =& M_t\sum_{i=1}^n dB_t^i.\end{aligned}$$

Integrating yields,

$$M_T = M_0 + \sum_{i=1}^{n} \int_0^T M_s dB_t^i.$$

Since $M_T(\omega) = \exp(-nT/2)F(\omega)$ and $M_0 = 1$ we get

$$F(\omega) = \exp(nT/2) + \sum_{i=1}^{n} \int_0^T \exp(n(T-s)/2) \exp\{B_t^1 + \cdots + B_t^n\} dB_s^i.$$

Hence, $z = \exp(nT/2)$ and $\varphi_i(s,\omega) = \exp(n(T-s)/2)\exp\{B_t^1 + \cdots + B_t^n\}$.

(b) Now we define

$$M_t = (B_t^1)^3 + \cdots + (B_t^n)^3 - 3tB_t^1 - \cdots - 3tB_t^n.$$

Then, $F(\omega) = M_T + 3T\sum_{i=1}^n B^i(T)$ and using Itô's formula we get

$$\begin{aligned} dM_t &= -3\sum_{i=1}^{n} B_t^i dt + 3\sum_{i=1}^{n} \big((B_t^i)^2 - t\big) dB_t^i + \frac{1}{2}\sum_{i=1}^{n} 6B_t^i dt \\ &= 3\sum_{i=1}^{n} \big((B_t^i)^2 - t\big) dB_t^i. \end{aligned}$$

Integration yields,

$$M_T = M_0 + 3\sum_{i=1}^{n} \int_0^T \big((B_s^i)^2 - s\big) dB_s^i.$$

Since $M_T(\omega) = F(\omega) - 3T\sum_{i=1}^n B_T^i$ and $M_0 = 0$ we get

$$\begin{aligned} F(\omega) &= 3T\sum_{i=1}^{n} B_T^i + 3\sum_{i=1}^{n} \int_0^T \big((B_s^i)^2 - s\big) dB_s^i \\ &= \sum_{i=1}^{n} \int_0^T 3(T - s + (B_s^i)^2) dB_s^i. \end{aligned}$$

Hence, $z = 0$ and $\varphi_i(s,\omega) = 3(T - s + (B_s^i)^2)$.

Exercise 15.5 Let Z_t solve the deterministic ODE

$$dZ_t = \alpha_t Z_t dt.$$

Then we know that

$$Z_t = Z_0 \exp(\int_0^t \alpha_s ds).$$

Let us therefore try to find a solution to the original equation of the form $X_t(\omega) = Y_t(\omega)Z_t$. Using the Itô formula to the function $f(y,z) = yz$ we get

$$\begin{aligned} dX_t &= Y_t dZ_t + Z_t dY_t + d\langle Y, Z\rangle_t \\ &= Y_t \alpha_t Z_t dt + Z_t dY_t = \alpha_t X_t + Z_t dY_t. \end{aligned}$$

Recalling that $dX_t = \alpha_t X_t dt + \beta_t X_t dB_t$ and $X_t = Y_t Z_t$ gives

$$\beta_t Y_t Z_t dB_t = Z_t dY_t.$$

We can identify

$$dY_t = \beta_t Y_t dB_t.$$

The solution to this SDE is the exponential martingale so we get

$$Y_t = Y_0 \exp\{\int_0^t \beta_s dB_s - \frac{1}{2}\int_0^t \beta_s^2 ds\}.$$

From this we see that

$$X_t = \exp\{\int_0^t \beta_s dB_s + \int_0^t (\alpha_s - \frac{1}{2}\beta_s^2) ds\},$$

since $X(0) = Z(0)Y(0) = 1$. To compute the expectation we know that

$$X_t = \int_0^t dX_t = \int_0^t \alpha_s X_s ds + \int_0^t \beta_s X_s dB_s.$$

Taking expectation on both sides yields,

$$\begin{aligned}
\mathbb{E}[X_t] &= \int_0^t \alpha_s \mathbb{E}[X_s] ds + \mathbb{E}[\int_0^t \beta_s X_s dB_s] \\
&= \int_0^t \alpha_s \mathbb{E}[X_s] ds.
\end{aligned}$$

If we denote $m_t = \mathbb{E}[X_t]$ we see that

$$m_t = \int_0^t \alpha_s m_s ds.$$

That is, m solves the differential equation

$$m_t' = \alpha_t m_t, \qquad m_0 = 1$$

Hence, $m_t = \exp(\int_0^t \alpha_s ds)$.

(b) Using results from (a) gives

$$Y_T = Y_0 + \int_0^T \beta_s Y_s dB_s$$

and

$$\begin{aligned}
X_T = Y_T Z_T &= [Y_0 + \int_0^T \beta_s Y_s dB_s] Z_0 \exp(\int_0^T \alpha_s ds) \\
&= Y_0 Z_0 e^{(\int_0^T \alpha_s ds)} + Y_0 Z_0 \int_0^T \beta_s e^{\int_0^s \beta_u dB_u - \frac{1}{2}\int_0^s \beta_u^2 du} dB_s e^{(\int_0^T \alpha_s ds)} \\
&= e^{(\int_0^T \alpha_s ds)} + \int_0^T e^{(\int_0^T \alpha_r dr)} \beta_s e^{\int_0^s \beta_u dB_u - \frac{1}{2}\int_0^s \beta_u^2 du} dB_s
\end{aligned}$$

Hence, $z = e^{(\int_0^T \alpha_s ds)}$ and $\varphi(s, \omega) = e^{(\int_0^T \alpha_r dr)} \beta_s e^{\int_0^s \beta_u dB_u - \frac{1}{2}\int_0^s \beta_u^2 du}$.

Exercise 15.6 Put $M_t = \int_0^{f(t)} \frac{1}{\sqrt{1+s}} dB_s$. We will use Theorem 15.4 on p. 116. Clearly M_t is a continuous martingale and it has quadratic variation

$$\langle M \rangle_t = \int_0^{f(t)} \frac{1}{1+s} ds = \log(1 + f(t)) = \frac{t^2}{2} = \int_0^t s ds.$$

Hence, there exists another Brownian motion $\tilde{B}$ such that

$$M_t = \int_0^t \sqrt{s} d\tilde{B}_s$$

Exercise 15.7 Put $M_t = \int_0^{f(t)} \sqrt{\frac{\arctan s}{1+s^2}} dB_s$. We will use Theorem 15.4 on p. 116. Clearly M_t is a continuous martingale and it has quadratic variation

$$\begin{aligned}
\langle M \rangle_t &= \int_0^{f(t)} \frac{\arctan s}{1+s^2} ds = \{u = \arctan s\} \\
&= \int_0^{\arctan f(t)} u du = \int_0^{t+2n\pi} u du = \frac{(t+2n\pi)^2}{2} = \int_0^t (s+2n\pi) ds,
\end{aligned}$$

for some $n = \dots, -1, 0, 1, \dots$. Hence, there exists another Brownian motion $\tilde{B}$ such that

$$M_t = \int_0^t \sqrt{s+2n\pi} d\tilde{B}_s$$

9 Chapter 16

Exercise 16.1 (a) Put

$$Y_t = S_t^{-1} = \exp\{(\frac{\sigma^2}{2} - \alpha)t - \sigma B_t\},$$

and let

$$g(t,x) = \exp\{(\frac{\sigma^2}{2} - \alpha)t - \sigma x\}.$$

Itô's formula yields,

$$dY_t = (\sigma^2 - \alpha) Y_t dt - \sigma Y_t dB_t.$$

(b) Itô's formula yields,

$$\begin{aligned}
d(X_t Y_t) &= X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t \\
&= X_t\big((\sigma^2 - \alpha) Y_t dt - \sigma Y_t dB_t\big) + Y_t\big((\alpha X_t + \beta) dt + (\sigma X_t + \gamma) dB_t\big) + (-\sigma Y_t(\sigma X_t + \gamma)) dt \\
&= (\beta - \sigma\gamma) Y_t dt + \gamma Y_t dB_t.
\end{aligned}$$

(c) Integrating the last equation yields,

$$\frac{X_t}{S_t} = \int_0^t \frac{\beta - \sigma\gamma}{S_r} dr + \int_0^t \frac{\gamma}{S_r} dB_r.$$

Hence,

$$X_t = S_t\Big(\int_0^t \frac{\beta - \sigma\gamma}{S_r} dr + \int_0^t \frac{\gamma}{S_r} dB_r\Big).$$

Exercise 16.2 The Markov property implies that

$$u(t, X_t) = \mathbb{E}[f(X_T^{X_t,t})] = \mathbb{E}[f(X_T) \mid X_t] = \mathbb{E}[f(X_T) \mid \mathcal{F}_t].$$

Now with $M_t = u(t, X_t)$ we find that

$$\begin{aligned}
\mathbb{E}[M_{t+h} \mid \mathcal{F}_t] &= \mathbb{E}[u(t+h, X_{t+h}) \mid \mathcal{F}_t] = \mathbb{E}\big[\mathbb{E}[f(X_T) \mid \mathcal{F}_{t+h}] \,\big|\, \mathcal{F}_t\big] \\
&= \mathbb{E}[f(X_T) \mid \mathcal{F}_t] = u(t, X_t) = M_t.
\end{aligned}$$

Hence, M_t has the martingale property. Since f is bounded $M_t = \mathbb{E}[f(X_T) \mid \mathcal{F}_t]$ is bounded almost surely and it follows that $\mathbb{E}[|M_t|] < \infty$. Therefore M_t is a martingale.

10 Chapter 17

Exercise 17.1 We have the general formula as in Theorem 17.2 on p. 140. In this exercise we are in the 1-dimensional case and we can identify the functions $b(t,x)$ and $\sigma(t,y)$ as

$$b(t,x) = \alpha + \theta x, \qquad \sigma(t,x) = \psi(x).$$

Hence, the infinitesimal generator is

$$Af(x) = (\alpha + \theta x)\frac{df}{dx} + \frac{1}{2}\psi^2(x)\frac{d^2 f}{dx^2}.$$

Exercise 17.2 We have the general formula as in Theorem 17.2 on p. 140. In this exercise we are in the 1-dimensional case and we can identify the functions $b(t,x)$ and $\sigma(t,y)$ as

$$b(t,y) = \theta\frac{y}{\sqrt{1+y^2}}, \qquad \sigma(t,x) = \sigma.$$

Hence, the infinitesimal generator is

$$Af(y) = \theta\frac{y}{\sqrt{1+y^2}}\frac{df}{dy} + \frac{1}{2}\sigma^2\frac{d^2 f}{dy^2}.$$

Exercise 17.3 (a) We have $R_t = \sqrt{(B_t^1)^2 + (B_t^2)^2 + (B_t^3)^2}$ where $B_t = (B_t^1, B_t^2, B_t^3)$ is a 3-dimensional Brownian motion. If we put $x = (x_1, x_2, x_3)$ and $r(x) = \sqrt{x_1^2 + x_2^2 + x_3^2}$. Then we see that $R_t = r(B_t)$. Hence, $f(R_t) = f(r(B_t))$ and we know that the infinitesimal generator for B_t is

$$\tilde{A}h(x_1, x_2, x_3) = \frac{1}{2}\sum_{j=1}^{3}\frac{\partial^2 h}{\partial x_j^2}.$$

With $h(x) = f(r(x))$ we get using the chain rule that

$$\frac{\partial^2 h(x)}{\partial x_j^2} = \frac{d^2 f}{dr^2}(r(x))\Big(\frac{\partial r}{\partial x_j}(x)\Big)^2 + \frac{df}{dr}(r(x))\frac{\partial^2 r}{\partial x_j^2}(x).$$

Thus, we get

$$\begin{aligned}
\tilde{A}h(x_1, x_2, x_3) &= \frac{1}{2}\sum_{j=1}^{3}\frac{\partial^2 h}{\partial x_j^2} \\
&= \frac{1}{2}\sum_{j=1}^{3}\frac{d^2 f}{dr^2}(r(x))\Big(\frac{\partial r}{\partial x_j}(x)\Big)^2 + \frac{df}{dr}(r(x))\frac{\partial^2 r}{\partial x_j^2}(x) \\
&= \frac{1}{2}\sum_{j=1}^{3}\frac{d^2 f}{dr^2}(r(x))\Big(\frac{x_j}{r(x)}\Big)^2 + \frac{df}{dr}(r(x))\frac{r(x) - x_j\frac{\partial r}{\partial x_j}(x)}{r(x)^2} \\
&= \frac{1}{2}\sum_{j=1}^{3}\frac{d^2 f}{dr^2}(r(x))\Big(\frac{x_j}{r(x)}\Big)^2 + \frac{df}{dr}(r(x))\frac{r(x) - \frac{x_j^2}{r(x)}}{r(x)^2} \\
&= \frac{1}{2}\Big(\frac{d^2 f}{dr^2}(r(x)) + \frac{2}{r(x)}\cdot\frac{df}{dr}(r(x))\Big).
\end{aligned}$$

Finally, since $R_t = r(B_t)$ we get the infinitesimal generator of R_t as

$$Af(r) = \frac{1}{2}\Big(\frac{d^2 f}{dr^2}(r) + \frac{2}{r} \cdot \frac{df}{dr}(r)\Big).$$

(b) Let us first derive the SDE satisfied by R_t. Using Itô's formula on $r(x)$ (this is possible even though r is not two times continuously differentiable at 0 because the probability of R_t hitting 0 is zero). We get

$$\begin{aligned}
dR_t = dr(B_t) &= \sum_{j=1}^{3} \frac{\partial r}{\partial x_j}(B_t) dB_t^j + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2 r}{\partial x_i \partial x_j}(B_t) d\langle B^i, B^j \rangle_t \\
&= \sum_{j=1}^{3} \frac{\partial r}{\partial x_j}(B_t) dB_t^j + \frac{1}{2} \sum_{j=1}^{3} \frac{\partial^2 r}{\partial x_j^2}(B_t) dt \\
&= \sum_{j=1}^{3} \frac{B_t^j}{r(B_t)} dB_t^j + \frac{1}{2} \sum_{j=1}^{3} \frac{r(B_t) - \frac{(B_t^j)^2}{r(B_t)}}{r^2(B_t)} dt \\
&= \sum_{j=1}^{3} \frac{B_t^j}{r(B_t)} dB_t^j + \frac{2}{2r(B_t)} dt.
\end{aligned}$$

Introduce the process $\tilde{B}_t = \sum_{j=1}^{3} \int_0^t \frac{B_s^j}{r(B_s)} dB_s^j$. Then we can write

$$dR_t = \frac{1}{R_t} dt + d\tilde{B}_t.$$

Now we can show that $\tilde{B}_t$ is in fact a Brownian motion! We use Lévy's characterization, Corollary 15.6 on p. 117 to prove this claim. First note that $\tilde{B}_t$ is a continuos martingale since it is a sum of Itô integrals which are themselves continuous martingales. Second, $\tilde{B}_t$ has the quadratic variation process

$$\begin{aligned}
\langle \tilde{B} \rangle_t &= \int_0^t d\langle \tilde{B} \rangle_s = \int_0^t \sum_{i,j=1}^{3} \frac{B_s^i B_s^j}{r^2(B_s)} d\langle B^i, B^j \rangle_s \\
&= \int_0^t \sum_{i=1}^{3} \frac{(B_s^i)^2}{r^2(B_s)} ds = \int_0^t 1\, ds = t.
\end{aligned}$$

Note that from the representation of R_t as the solution to

$$dR_t = \frac{1}{R_t} dt + d\tilde{B}_t,$$

we can easily derive the expression for the infinitesimal generator in (a).

Let us continue to prove that $Z_t = \frac{\sinh(R_t)}{R_t} e^{-t/2}$ is a martingale. Put $f(t,r) = \frac{\sinh(r)}{r} e^{-t/2}$. The partial derivatives of f can be computed as

$$\begin{aligned}
\frac{\partial f}{\partial t} &= -\frac{1}{2} f(t,r), \qquad \frac{\partial f}{\partial r} = e^{-t/2}\big(\frac{\cosh(r)}{r} - \frac{\sinh(r)}{r^2}\big), \\
\frac{\partial^2 f}{\partial r^2} &= e^{-t/2}\big(\frac{\sinh(r)}{r} - 2\frac{\cosh(r)}{r^2} + 2\frac{\sinh(r)}{r^3}\big).
\end{aligned}$$

Combining these expressions we see that

$$\frac{\partial f(t,r)}{\partial t} + \frac{1}{r} \cdot \frac{\partial f(t,r)}{\partial r} + \frac{1}{2} \cdot \frac{\partial^2 f(t,r)}{\partial r^2} = 0.$$

Applying Itô's formula we get

$$\begin{aligned}
df(t,R_t) &= \frac{\partial f(t,R_t)}{\partial t}dt + \frac{\partial f(t,R_t)}{\partial r}dR_t + \frac{1}{2}\cdot\frac{\partial^2 f(t,R_t)}{\partial r^2}d\langle R\rangle_t \\
&= \Big(\frac{\partial f(t,R_t)}{\partial t} + \frac{1}{R_t}\cdot\frac{\partial f(t,R_t)}{\partial r} + \frac{1}{2}\cdot\frac{\partial^2 f(t,R_t)}{\partial r^2}\Big)dt + \frac{\partial f(t,R_t)}{\partial r}d\tilde{B}_t \\
&= \frac{\partial f(t,R_t)}{\partial r}d\tilde{B}_t.
\end{aligned}$$

Integrating yields,

$$Z_t - Z_0 = f(t,R_t) - f(0,R_0) = \int_0^t \frac{\partial f(s,R_s)}{\partial r}d\tilde{B}_s,$$

which is an Itô integral and hence Z_t is a martingale.

Exercise 17.5 (a) We have the general formula as in Theorem 17.2 on p. 140. In this exercise we are in the 1-dimensional case and we can identify the functions $b(t,x)$ and $\sigma(t,x)$ as

$$b(t,x) = \mu x, \qquad \sigma(t,x) = \sigma x^\gamma.$$

Hence, the infinitesimal generator is

$$Af(x) = \mu x\frac{df}{dx} + \frac{1}{2}\sigma^2 x^{2\gamma}\frac{d^2 f}{dx^2}.$$

(b) We have $X_t = \log S_t$ or equivalently $S_t = e^{X_t}$. Using Itô's formula on $g(x) = \log x$ we get

$$\begin{aligned}
dX_t &= \frac{1}{S_t}dS_t + \frac{1}{2}\cdot\big(-\frac{1}{S_t^2}\big)d\langle S\rangle_t \\
&= \mu dt + \sigma S_t^{\gamma-1}dB_t - \frac{1}{2}\sigma^2 S_t^{2(\gamma-1)}dt \\
&= \big(\mu - \frac{1}{2}\sigma^2 S_t^{2(\gamma-1)}\big)dt + \sigma S_t^{\gamma-1}dB_t \\
&= \big(\mu - \frac{1}{2}\sigma^2 e^{2(\gamma-1)X_t}\big)dt + \sigma e^{(\gamma-1)X_t}dB_t.
\end{aligned}$$

Exercise 17.6 (a) We need to show that

$$\int_0^t e^{\beta(t-s)}dX_s = X_t + \beta\int_0^t e^{\beta(t-s)}X_s ds,$$

or equivalently that

$$e^{-\beta t}X_t = \int_0^t e^{-\beta s}dX_s - \beta\int_0^t e^{-\beta s}X_s ds.$$

Itô's formula applied on $e^{-\beta t}X_t$ gives,

$$d(e^{-\beta t}X_t) = -\beta e^{-\beta t}X_t dt + e^{-\beta t}dX_t,$$

and integrating yields the desired result.
(b) We have

$$e^{-\beta t}Y_t = x + \int_0^t e^{-\beta s}dX_s,$$

so applying Itô's formula to $e^{-\beta t}Y_t$ gives

$$d(e^{-\beta t}Y_t) = -\beta e^{-\beta t}Y_t dt + e^{-\beta t}dY_t.$$

On the other hand we also have

$$d(e^{-\beta t}Y_t) = e^{-\beta t}dX_t = e^{-\beta t}dB_t + ce^{-\beta t}dt.$$

Putting these two expressions equal and solving for dY_t yields

$$dY_t = (\beta Y_t + c)dt + dB_t.$$

Hence, by applying Theorem 17.2 on p. 140 we identify the infinitesimal generator of Y_t as

$$Af(y) = (\beta y + c)\frac{df}{dy}(y) + \frac{d^2 f}{dy^2}(y).$$

(c) Since we have $dX_s = dB_s + cds$ we see that

$$Y_t = e^{\beta t}x + \int_0^t e^{\beta(t-s)}dB_s + c\int_0^t e^{\beta(t-s)}ds.$$

We know that

$$e^{\beta(t-s)}dB_s \sim N\big(0, \int_0^t e^{2\beta(t-s)}ds\big),$$

which implies that

$$Y_t \sim N\big(e^{\beta t}x + c\int_0^t e^{\beta(t-s)}ds, \int_0^t e^{2\beta(t-s)}ds\big).$$

Exercise 17.7 (a) We have the general formula as in Theorem 17.2 on p. 140. In this exercise we are in the 1-dimensional case and we can identify the functions $b(t,x)$ and $\sigma(t,x)$ as

$$b(t,x) = \mu(x), \qquad \sigma(t,x) = \sigma x(1-x).$$

Hence, the infinitesimal generator is

$$Af(x) = \mu(x)\frac{df}{dx} + \frac{1}{2}\sigma^2 x^2(1-x)^2\frac{d^2 f}{dx^2}.$$

(b) The SDE to solve is

$$dX_t = -X_t^2(1-X_t)dt + X_t(1-X_t)dB_t.$$

Let $g(x) = x(1-x)$. Then $g'(x) = 1-2x$ and we get

$$\frac{1}{2}g(x)g'(x) = \frac{1}{2}x(1-x)(1-2x)$$

Now, since

$$\frac{1}{2}g(x) + \frac{1}{2}g(x)g'(x) = -x^2(1-x)$$

we can use the first extension of the Doss and Sussman technique. The solution to the SDE is therefore given by

$$X_t = h^{-1}\big(-\frac{1}{2}t + B_t\big),$$

where $h(x) = \int_{x_0}^{x} \frac{1}{g(y)} dy$. It remains to determine $h^{-1}(x)$.

$$\begin{aligned} h(x) &= \int_{x_0}^{x} \frac{1}{y(1-y)} dy = \int_{x_0}^{x} \big(\frac{1}{y} - \frac{1}{1-y}\big) dy = \log\big(\frac{x}{x_0}\big) - \log\big(\frac{x-1}{x_0-1}\big) \\ &= \log\big(\frac{1-x_0}{x_0} \cdot \frac{x}{1-x}\big). \end{aligned}$$

From this we get

$$\frac{x}{1-x} = \frac{1-x_0}{x_0} e^{h(x)},$$

from which it follows that

$$h^{-1}(x) = \frac{x_0 e^x}{1 - x_0 + x_0 e^x}.$$

Using this we can write the solution as

$$X_t = \frac{x_0 e^{-t/2 + B_t}}{1 - x_0 + x_0 e^{-t/2+B_t}}.$$

Exercise 17.8 To determine the probability law we negin by solving the SDE. First note that the n-dimensional SDE is nothing but n independent SDE's.

$$dX_t^i = -\frac{1}{2}\beta_t X_t^i dt + \frac{1}{2}\sigma_t dB_t^i, \qquad i = 1, \dots, n.$$

To solve these SDE's we consider the process $X_t^i e^{\frac{1}{2}\int_0^t \beta_s ds}$. Applying Itô's formula yields,

$$d\big(X_t^i e^{\frac{1}{2}\int_0^t \beta_s ds}\big) = \frac{1}{2}\beta_t e^{\frac{1}{2}\int_0^t \beta_s ds} X_t^i dt + e^{\frac{1}{2}\int_0^t \beta_s ds} dX_t^i.$$

Inserting the expression for dX_t^i and integrating gives

$$X_t^i e^{\frac{1}{2}\int_0^t \beta_s ds} = x_i + \frac{1}{2}\int_0^t e^{\frac{1}{2}\int_0^s \beta_u du} \sigma_s dB_s^i,$$

or equivalently

$$X_t^i = e^{\frac{1}{2}\int_0^t \beta_s ds}\big(x_i + \frac{1}{2}\int_0^t e^{\frac{1}{2}\int_0^s \beta_u du} \sigma_s dB_s^i\big).$$

We know that if $f(x)$ is a deterministic function with $\int_0^t f^2(s)ds < \infty$, then

$$\int_0^t f(s) dB_s \sim N(0, \int_0^t f^2(s)ds).$$

It follows since all the SDE's are independent that

$$X_t \sim N(xe^{-\frac{1}{2}\int_0^t \beta_s ds}, \Sigma)$$

where the covariance matrix Σ is diagonal with

$$\Sigma_{ii} = \frac{1}{4} e^{-\int_0^t \beta_s ds} \int_0^t e^{\int_0^s \beta_u du} \sigma_s^2 ds.$$

(b) We have

$$r_t = \|X_t\|^2 = (X_t^1)^2 + \cdots + (X_t^n)^2.$$

Using Itô's formula gives

$$\begin{aligned} dr_t &= \sum_{i=1}^n 2X_t^i dX_t^i + \frac{1}{2}\sum_{i=1}^n 2d\langle X^i\rangle_t \\ &= -\beta_t \sum_{i=1}^n (X_t^i)^2 dt + \sigma_t \sum_{i=1}^n X_t^i dB_t^i + \frac{1}{4}\sum_{i=1}^n \sigma_t^2 dt \\ &= \big(\frac{\sigma_t^2}{4} - \beta_t r_t\big)dt + \sigma_t \sum_{i=1}^n X_t^i dB_t^i. \end{aligned}$$

Now we observe that

$$\langle \int_0^\cdot \sum_{i=1}^n X_s^i dB_s \rangle_t = \int_0^t \sum_{i=1}^n (X_s^i)^2 ds = \int_0^t r_s ds.$$

So by Theorem 15.4 on p. 116 there exists a Brownian motion $\tilde{B}_t$ such that

$$\int_0^t \sum_{i=1}^n X_s^i dB_s \stackrel{\mathrm{d}}{=} \int_0^t \sqrt{r_s} d\tilde{B}_s.$$

This gives the Cox-Ingersoll-Ross model

$$dr_t = \big(\frac{\sigma_t^2}{4} - \beta_t r_t\big)dt + \sigma_t \sqrt{r_t} d\tilde{B}_t.$$

Exercise 17.9 Using the integration by parts formula on $(X_t^1)^2$, $X_t^1 X_t^2$ and $(X_t^2)^2$ gives

$$\begin{cases} (X_t^1)^2 &= x_1^2 + 2\int_0^t X_s^1 X_s^2 ds, \\ X_t^1 X_t^2 &= x_1 x_2 - \int_0^t (X_s^1)^2 ds + c\int_0^t (X_s^1)^2 dB_s + \int_0^t (X_s^2)^2 ds, \\ (X_t^2)^2 &= x_2^2 - 2\int_0^t X_s^1 X_s^2 ds + 2c\int_0^t X_s^1 X_s^2 dB_s + c^2 \int_0^t (X_s^1)^2 ds. \end{cases}$$

Taking expectation of each equation and using the fact that Itô integrals have expectation 0 gives us,

$$\begin{cases} m_1(t) &= x_1^2 + 2\int_0^t m_2(s) ds, \\ m_2(t) &= x_1 x_2 - \int_0^t m_1(s) ds + \int_0^t m_3(s) ds, \\ m_3(t) &= x_2^2 - 2\int_0^t m_2(s) ds + c^2 \int_0^t m_1(s) ds. \end{cases}$$

Differentiation finally yields with $m(t) = (m_1(t), m_2(t), m_3(t))^T$,

$$\frac{dm}{dt}(t) = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ c^2 & -2 & 0 \end{pmatrix} m(t), \qquad m(0) = \begin{pmatrix} x_1^2 \\ x_1 x_2 \\ x_2^2 \end{pmatrix}.$$

11 Chapter 18

Exercise 18.1 We use the Feynman-Kac formula with $f(x) = 1$ and let X_t be an n-dimensional Brownian motion B_t. We know that the infinitesimal generator of an n-dimensional Brownian motion is given by $A = \frac{1}{2}\Delta$ where Δ is the n-dimensional Laplace operator. It follows from the Feynman-Kac theorem (Theorem 18.5 p. 149) that $u(t,x)$ solves

$$\begin{cases} \frac{\partial u}{\partial t} & = \frac{1}{2}\Delta u - qu, \quad t > 0,\ x \in \mathbb{R}^n \\ u(x,0) & = 1, \qquad\qquad\quad x \in \mathbb{R}^n. \end{cases}$$

Using the PDE above and $u(t,x) = e^{-V(t,x)}$, $V(t,x) = -\log(u(t,x))$ gives

$$\frac{\partial V}{\partial t} = -\frac{1}{u} \cdot \frac{\partial u}{\partial t} = -\frac{1}{u}\big(\frac{1}{2}\Delta u - qu\big) = -\frac{1}{2}e^V\Delta(e^{-V}) + q$$

Let us now compute

$$\Delta(e^{-V(t,x)}) = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} e^{-V(t,x)},$$

where $x = (x_1, \dots, x_n)$. We have for $i = 1, \dots, n$

$$\frac{\partial^2}{\partial x_i^2} e^{-V(t,x)} = \frac{\partial}{\partial x_i}\big(- \frac{\partial V}{\partial x_i} e^{-V}\big) = \frac{\partial^2 V}{\partial x_i^2} e^{-V} + \big(\frac{\partial V}{\partial x_i}\big)^2 e^{-V},$$

and hence

$$\Delta e^{-V} = e^{-V}\big((\nabla V)^2 - \Delta V\big),$$

where $(\nabla V)^2 = \nabla V \cdot \nabla V$ is the inner product of ∇V with itself. Note now that $V(0,x) = 0$ for $x \in \mathbb{R}^n$. This gives the PDE for $V(t,x)$

$$\begin{cases} \frac{\partial V}{\partial t} & = \Delta V - (\nabla V)^2 + q, \\ V(x,0) & = 0, \qquad x \in \mathbb{R}^n. \end{cases}$$

Exercise 18.2 Using the general case of the Feynman-Kac formula we can identify

$$q(x) = -V(x), \qquad g(x) = g(x),$$

and we arrive at the following PDE for $u(t,x)$

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) & = Au(t,x) + V(x)u(t,x) + g(x), \\ u(0,x) & = f(x). \end{cases}$$

Exercise 18.3 Assume X_t is the solution the the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \qquad X_0 = x_0.$$

Using Itô's formula on $M_t = \varphi(t, X_t)$ we get

$$\begin{aligned} dM_t = d\varphi(t,X_t) &= \frac{\partial \varphi}{\partial t}(t,X_t)dt + \frac{\partial \varphi}{\partial x}(t,X_t)dX_t + \frac{1}{2} \cdot \frac{\partial^2 \varphi}{\partial x^2}(t,X_t)d\langle X\rangle_t \\ &= \frac{\partial \varphi}{\partial t}(t,X_t)dt + \frac{\partial \varphi}{\partial x}(t,X_t)dX_t + \frac{1}{2} \cdot \frac{\partial^2 \varphi}{\partial x^2}(t,X_t)\sigma^2(t,X_t)dt \\ &= \big(\frac{\partial \varphi}{\partial t}(t,X_t) + \frac{1}{2} \cdot \frac{\partial^2 \varphi}{\partial x^2}(t,X_t)\sigma^2(t,X_t)\big)dt + \frac{\partial \varphi}{\partial x}(t,X_t)dX_t \\ &= \frac{\partial \varphi}{\partial x}(t,X_t)dX_t, \end{aligned}$$

if $\frac{\partial \varphi}{\partial t}(t, X_t) + \frac{1}{2} \cdot \frac{\partial^2 \varphi}{\partial x^2}(t, X_t)\sigma^2(t, X_t) = 0$. Computing the partial derivatives yields

$$\frac{\partial \varphi}{\partial t}(t, x) = \frac{\varphi(t, x)}{2(1-t)}\big(1 - \frac{x^2}{1-t}\big), \qquad \frac{\partial^2 \varphi}{\partial x^2}(t, x) = -\frac{\varphi(t, x)}{(1-t)}\big(1 - \frac{x^2}{1-t}\big)$$

Hence, $\frac{\partial \varphi}{\partial t}(t, X_t) + \frac{1}{2} \cdot \frac{\partial^2 \varphi}{\partial x^2}(t, X_t)\sigma^2(t, X_t) = 0$ if $\sigma(t, X_t) = 1$. It follows that X_t must be of the form

$$dX_t = b(t, X_t)dt + dB_t, \qquad X_0 = x_0.$$