# SF2970: Repetition 1

Spring 2017

## Conditional expectation

**Q 1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a $\sigma$-subalgebra.

a.) Let $X$ be an integrable random variable (i.e., $\mathbb{E}|X| < \infty$). Give the definition of $\mathbb{E}[X|\mathcal{G}]$.

b.) Let $Y$ be another integrable random variable. Assume now that $X$ is independent of $\mathcal{G}$ and that $Y$ is $\mathcal{G}$-measurable. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Borel function. How can you compute $\mathbb{E}[h(X,Y)|\mathcal{G}]$?

**Q 2.** Let $X_1, \dots, X_n$ be iid random variables. Set $S_i := \sum_{k=1}^{i} X_k$, $1 \le i \le n$. Compute $\mathbb{E}[S_n|\sigma(S_i)]$.

**Q 3.** Let $A$ and $B$ be two random variables whose joint distribution is uniform on the triangle $\{(a,b) \in \mathbb{R}^2 : 0 \le b \le a \le 1\}$. Compute the distribution of $B/A$. Show that $B/A$ and $A$ are independent. Compute the conditional expectation $\mathbb{E}[B|\sigma(A)]$.

## Martingales in discrete time

**Q 4.** a.) Let $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, \mathbb{P})$ be a filtered probability space. Let $M_n$, $n = 0, 1, 2, \dots$, be an $(\underline{\mathcal{F}}, \mathbb{P})$-martingale. Show that $\mathbb{E}M_n = \mathbb{E}M_0$, for all $n \ge 0$.

b.) An urn initially contains one blue and one yellow ball. At each discrete time step $n = 1, 2, \dots$, we select a ball from the urn (uniformly at random) and replace it with two balls of the same color. Let $(Y_n)$, $n = 1, 2 \dots$ be the number of yellow balls, with $Y_1 = 1$. Show that

$$M_n := \frac{Y_n}{n+1}$$

is a martingale in the natural filtration. Moreover, compute the distribution of $M_n$. (Hint: Prove by induction that $\mathbb{P}(Y_n = k) = 1/n$, for $k = 1, \dots, n$.)

**Q 5.** Let $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, \mathbb{P})$ be a filtered probability space. Let $M_n$, $n = 0, 1, 2, \dots$ be a square integrable martingale.

a.) Show that $M_n^2$ is a sub-martingale, i.e., show that $\mathbb{E}[M_{n+1}^2|\mathcal{F}_n] \ge M_n^2$, $n = 0, 1, 2, \dots$.

b.) Show that $\mathrm{cov}(M_{n+1} - M_n, M_n) = 0$, $n = 0, 1, 2, \dots$, where $\mathrm{cov}(X,Y) := \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$, for arbitrary random variables $X$ and $Y$. Thus $M_{n+1} - M_n$ and $M_n$ are uncorrelated (but not necessarily independent).

c.) The quadratic variation of $M_n$ is defined as

$$\langle M \rangle_n := \sum_{i=0}^{n-1} \mathbb{E}[(M_{i+1} - M_i)^2|\mathcal{F}_i]\,, \qquad n \ge 1\,,$$
$$\langle M \rangle_0 := 0\,.$$

Argue that $\langle M \rangle_n$ is $\mathcal{F}_{n-1}$ measurable. Show that $M_n^2 - \langle M \rangle_n$, $n = 0, 1, 2, \dots$, is a martingale, i.e,

$$\mathbb{E}[M_{n+1}^2 - \langle M \rangle_{n+1}|\mathcal{F}_n] = M_n^2 - \langle M \rangle_n\,.$$

## Discrete stochastic integral

**Q 6.** In the lecture we considered a simple discrete time model of a market with a stock and a bond. The stock price $(S_n)_{n\geq 0}$ was assumed to be given by

$$S_n = \exp\Big[\Big(\mu - \frac{1}{2}\sigma^2\Big)n + \sigma W_n\Big] S_0\,,$$

with $S_0 > 0$ the deterministic initial price, and constants $\mu \in \mathbb{R}$ and $\sigma > 0$. Here, $(W_n)$ is a standard discrete Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{P}$ the "market measure".

The bond's price $(B_n)_{n\geq 0}$ is assumed to be deterministic and given by

$$B_n = \mathrm{e}^{nr} B_0\,,$$

with $B_0 > 0$ the deterministic initial price, and $r \in \mathbb{R}$ the bond's rate.

a.) Fix some (large) final time $N \in \mathbb{N}$. Using Girsanov's theorem, we showed in class that $S_n$ can be written as

$$S_n = \exp\Big[\Big(r - \frac{1}{2}\sigma^2\Big)n + \sigma \widetilde{W}_n\Big] S_0\,, \qquad 0 \leq n \leq N\,,$$

with $\widetilde{W}_n$ a discrete Brownian motion under the "risk-neutral" measure $\widetilde{\mathbb{P}}$. Give the definition of $\widetilde{\mathbb{P}}$ in terms of $\mathbb{P}$, $W_N$ and $\theta := (\mu - r)/\sigma$.

Consider now a self-financing portfolio. Let $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ be the number of stocks respectively bonds bought at times $n$ and hold over the time intervals $[n, n+1)$. We assume that $(X_n)_{n\geq 0}$ and $(Y_n)_{n\geq 0}$ are bounded $\underline{\mathcal{F}}$-adapted stochastic processes, where $\underline{\mathcal{F}}$ is the filtration generated by $(W_n)_{n\geq 0}$, i.e., $\underline{\mathcal{F}} := (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, \ldots)$ with $\mathcal{F}_n = \sigma(\widetilde{W}_i : 0 \leq i \leq n) = \sigma(W_i : 0 \leq i \leq n)$.

The value of the portfolio over the time intervals $[n, n+1)$ is

$$V_n := X_n S_n + Y_n B_n\,, \qquad 1 \leq n \leq N\,,$$

with $V_0$ its initial value.

b.) What does it mean that the portfolio is self-financing?

c.) Show that $V_n$ can be written as

$$V_n = I_S(X)_n + I_B(Y)_n + V_0\,, \qquad 1 \leq n \leq N\,,$$

where $I_S(X)$ is the discrete stochastic integral of $(X_n)$ against $(S_n)$, etc. Are $(I_S(X)_n)$ and $(I_B(Y)_n)$ $(\underline{\mathcal{F}}, \widetilde{\mathbb{P}})$-martingales? (Warning: Here, $(B_n)$ is the bond price and not a discrete Brownian motion.)

In class we showed that the discounted portfolio value $\widetilde{V}_n := V_n/B_n$ and the discounted stock price $\widetilde{S}_n := S_n/B_n$ are both $(\underline{\mathcal{F}}, \widetilde{\mathbb{P}})$-martingales for $0 \leq n \leq N$.

d.) Find the adapted bounded process $(L_n)_{n\geq 0}$ such that

$$\widetilde{V}_n = I_{\widetilde{S}}(L)_n + \widetilde{V}_0\,, \qquad 1 \leq n \leq N\,.$$

Is $(I_{\widetilde{S}}(L)_n)$ an $(\underline{\mathcal{F}}, \widetilde{\mathbb{P}})$-martingale for $0 \leq n \leq N$?