## SF2970: Repetition 2

Spring 2017

### Brownian motion

**Q 1.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $B_t$ be a standard Brownian motion.

a.) Compute

$$\mathbb{E}[B_t B_s B_u]\,, \qquad t, s, u \geq 0\,.$$

b.) Let $c > 0$. Show that

$$Y_t := cB_{t/c^2}\,, \qquad t \geq 0\,,$$

is a standard Brownian motion. (In what filtration?)

**Q 2.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $B_t = (B_t^{(1)}, B_t^{(2)})$ be a standard 2-dimensional Brownian motion started at zero. Fix $r > 0$ and let

$$D_r := \{(x, y) \in \mathbb{R}^2 \, : x^2 + y^2 \leq r^2\}\,.$$

Compute $\mathbb{P}(B_t \in D_r)$, $t \geq 0$.

### Stochastic integrals

**Q 3.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $B_t = (B_t^{(1)}, B_t^{(2)})$ be a standard 2-dimensional Brownian motion started at zero. Fix $T > 0$ and set

$$f(t, \omega) = (f^{(1)}(t, \omega), f^{(2)}(t, \omega)) := (\mathbb{1}_{\{B_t^1 < B_t^2\}}, |B_t^{(1)} - B_t^{(2)}|^{1/2})\,, \qquad 0 \leq t \leq T\,.$$

Are $f^1(t, \omega)$ and $f^2(t, \omega)$ in $\mathcal{V}[0, T]$? Define

$$K_t := \int_0^t f^{(1)}(s, \omega)\mathrm{d}B_s^{(1)}(\omega) + \int_0^t f^{(2)}(s, \omega)\mathrm{d}B_s^{(2)}(\omega)\,, \qquad 0 \leq t \leq T\,.$$

Compute the expected value of the bracket process $\langle K \rangle_t$.

### Ito Lemma

**Q 4.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $B_t$ be a standard Brownian motion. Compute the stochastic differentials of the following processes:

$$\begin{aligned}
X_t &= \mathrm{e}^{\alpha t + \beta B_t}\,, && \alpha, \beta \in \mathbb{R}\,;\\
X_t &= \mathrm{e}^{\alpha t/2} \sin(\beta B_t)\,, && \alpha, \beta \in \mathbb{R}\,;\\
X_t &= \big(\alpha^{1/3} + \tfrac{1}{3} B_t\big)^3\,, && \alpha > 0\,;\\
X_t &= (B_t + t)\mathrm{e}^{-B_t - \frac{1}{2}t}\,;\\
X_t &= (B_t^2 + 1)^{-1}\,.
\end{aligned}$$

**Q 5.** Let $(\Omega, \mathcal{F}, \underline{\mathcal{F}}, \mathbb{P})$ be a filtered probability space. Let $B_t$ be an $\underline{\mathcal{F}}$-Brownian motion.

a.) Is

$$X_t = \mathrm{e}^{-t/2} \cosh(B_t)\,, \qquad t \geq 0\,,$$

an $\underline{\mathcal{F}}$-martingale? (Recall that $\cosh(x) = \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})$.)

b.) Find $f \, : \mathbb{R} \to \mathbb{R}$ such that $Y_t := f(B_t + t)$ is an $\underline{\mathcal{F}}$-martingale.