Stochastic Calculus An Introduction with Applications Problems with Solution

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## Chapters 1 to 4

## 4.1

Show that if A and B belongs to the  $\sigma$ -algebra  $\mathcal{F}$  then also  $B \setminus A \in \mathcal{F}$  (for definition of  $\sigma$ -algebra, see Definition 1.3). Also show that  $\mathcal{F}$  is closed under countable intersections, i.e. if  $A_i \in \mathcal{F}$  for  $i = 1, 2, \ldots$ , then  $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$ .

*Proof.* 1)  $B \setminus A = B \cap A^c = (B^c \cup A)^c$  and since  $B^c \in \mathcal{F}, B^c \cup A \in \mathcal{F}$  so  $(B^c \cup A)^c \in \mathcal{F}$ .

2) Take  $A_i \in \mathcal{F}$  for i = 1, 2, ... Since  $\bigcup_{i=1}^{\infty} A^c \in \mathcal{F}$  it follows that  $(\bigcup_{i=1}^{\infty} A^c)^c = \bigcap_{i=1}^{\infty} A \in \mathcal{F}$ 

## 4.6

Though a fair die once. Assume that we only can observe if the number obtained is "small",  $A = \{1, 2, 3\}$  and if the number is odd,  $B = \{1, 3, 5\}$ . Describe the resulting probability space; in particular, describe the  $\sigma$ -algebra  $\mathcal{F}$  generated by A and B in terms of a suitable partition (for definition of a partition, see Definition 1.9) of the sample space.

*Proof.* Looking at the Venn diagram **insert diagram**, we conclude that there are at most four partitions of the space,  $A \cap B = \{1,3\}, A \cap B^c = \{2\}, A^c \cap B = \{5\}$  and  $(A \cup B)^c = \{4,6\}$  of which none is an empty sets. These partitions can be combined in  $2^4 = 16$  different ways to generate the  $\sigma$ -algebra  $\mathcal{F}$  defined below.

$$\begin{split} \mathcal{F} = & \{ \emptyset, \Omega, A, B, A^c, B^c, A \cup B, A \cup B^c, A^c \cup B, A^c \cup B^c, \\ & A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c, (A \cap B^c) \cup (A^c \cap B), (A \cap B) \cup (A^c \cap B^c) \} \\ = & \{ \emptyset, \Omega, \{1, 2, 3\}, \{1, 3, 5\}, \{4, 5, 6\}, \{2, 4, 6\}, \{1, 2, 3, 5\}, \{1, 2, 3, 4, 6\}, \\ & \{1, 3, 4, 5, 6\}, \{2, 4, 5, 6\}, \{1, 3\}, \{2\}, \{5\}, \{4, 6\}, \{2, 5\}, \{1, 3, 4, 6\} \}. \end{split}$$

The probability measure on each of the sets in  $\mathcal{F}$  may be deduced by using the additivity of the probability measure and the probability measure of each of the partitions,  $P(A \cap B) = 2/6$ ,  $P(A \cap B^c) = 1/6$ ,  $P(A^c \cap B) = 1/6$  and  $P((A \cup B)^c) = 2/6$ .

### 4.10

Given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and functions  $X : \Omega \to \mathbb{R}, Y : \Omega \to \mathbb{R}$ . Define  $Z = \max\{X, Y\}$ .

- 1. Show that Z is  $\mathcal{F}$ -measurable if both X and Y are  $\mathcal{F}$ -measurable.
- 2. Find a special case where Z is  $\mathcal{F}$  measurable even though neither X nor Y is.

Proof. 1)

$$\{\omega \in \Omega : Z(\omega) \le x\} = \{\omega \in \Omega : \max\{X(\omega), Y(\omega)\} \le x\}$$
$$= \underbrace{\{\omega \in \Omega : X(\omega) \le x\}}_{\in \mathcal{F}} \cap \underbrace{\{\omega \in \Omega : X(\omega) \le x\}}_{\in \mathcal{F}} \in \mathcal{F},$$

where the last conclusion is based on the result of proposition 3.2.

2) Suppose you threw two coins and our  $\sigma$ -algebra  $\mathcal{F}$  was generated by the single set  $A = \{HT, TH\}$ , i.e.  $\mathcal{F} = \{\emptyset, \Omega, A, A^c\} = \{\emptyset, \{HH, TT, HT, TH\}, \{HT, TH\}, \{HH, TT\}\}$ . Let

$$X = \left\{ \begin{array}{ll} 1 & \text{if} \quad \omega \in \{HT\} \\ 0 & \text{if} \quad \omega \in \{HH, TT, TH\} \end{array} \right\} Y = \left\{ \begin{array}{ll} 1 & \text{if} \quad \omega \in \{TH\} \\ 0 & \text{if} \quad \omega \in \{HH, TT, HT\} \end{array} \right.$$

Then

$$Z = \begin{cases} 1 & \text{if } \omega \in \{HT, TH\} \\ 0 & \text{if } \omega \in \{HH, TT\} \end{cases}$$

hence X and Y are not  $\mathcal{F}$ -measurable while Z is.

## 4.11

Toss a fair soin n = 4 times. describe the sample space  $\Omega$ . we want to consider functions  $X : \omega \to \{-1, 1\}$ .

- 1. describe the probability space  $(\Omega, \mathcal{F}_1, \mathbb{P}_1)$  that arises if we want each outcome to be a legitimate event. How many  $\mathcal{F}_1$ -measurable functions X with  $\mathbb{E}[X] = 0$  are there?
- 2. Now describe the probability space  $(\Omega, \mathcal{F}_2, \mathbb{P}_2)$  that arises if we want that only combinations of sets of the type

$$A_i = \{ \omega \in \Omega : \text{ Number of heads } = i \}, \quad i = \{0, 1, 2, 3, 4\}$$

should be events. How many  $\mathcal{F}_2$ -measurable functions X with  $\mathbb{E}[X] = 0$  are there?

3. Solve 2) above for some other n.

*Proof.* 1) Each coin that is tossed has to possible outcomes, and since there are four coins to be tossed, there are  $2^4 = 16$  possible outcomes, or partitions. So the sample space is  $\Omega = \{HHHH, HHHT, HHTH, etc.\}$  with 16 members that describes all the information from that four tosses.  $\mathcal{F}_1$  is the *power set* of  $\Omega$  consisting of all combinations of sets of  $\Omega$  (don't forget that  $\emptyset$  allways is included in a  $\sigma$ -algebra ), hence  $\mathcal{F}_1 = \sigma(\Omega)$ .  $\mathbb{P}_1$  is deduced by first observing that each  $\omega \in \Omega$  has  $\mathbb{P}(\omega) = 1/16$  and then using the additivity och the probability measure.

For any function  $X : \Omega \to \{-1, 1\}$  that is measurable with respect to  $\mathcal{F}_1$  there is a set  $A \in \mathcal{F}_1$  such that

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ -1 & \text{if } \omega \in A^c. \end{cases}$$

Hence  $\mathbb{E}[X] = \mathbb{P}_1(A) - \mathbb{P}_1(A^c)$  so if  $\mathbb{E}[X] = 0$  we must have  $\mathbb{P}_1(A) = \mathbb{P}_1(A^c)$ , and since there are 16 partitions each with probability measure 1/16, there are <sup>16</sup>/<sub>8</sub> ways of combining the partitions so that there are equally many of them in A and  $A^c$ , hence there are <sup>16</sup>/<sub>8</sub> possible functions  $X : \Omega \to \{-1, 1\}$  such that  $\mathbb{E}[X] = 0$ . 2) The sets

$$A_{0} = \{TTTT\}$$

$$A_{1} = \{TTTH, TTHT, THTT, HTTT\}$$

$$A_{2} = \{TTHH, THHT, HHTT, HTTH, THTH, HTHT\}$$

$$A_{3} = \{HHHT, HHTH, HTHH, THHH\}$$

$$A_{4} = \{HHHH\},$$

defines a partition of  $\Omega$ . The  $\sigma$ -algebra  $\mathcal{F}_2$  is the  $\sigma$ -algebra generated by this partition. Define  $\mathbb{P}_i = \mathbb{P}(A_i)$ , then  $\mathbb{P}_0 = 1/16$ ,  $\mathbb{P}_1 = 4/16$ ,  $\mathbb{P}_2 = 6/16$ ,  $\mathbb{P}_3 = 4/16$  and  $\mathbb{P}_4 = 1/16$ . For any function  $X : \Omega \to \{-1, 1\}$  that is measurable with respect to  $\mathcal{F}_2$  there is a set  $A \in \mathcal{F}_2$  such that

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ -1 & \text{if } \omega \in A^c \end{cases}$$

Hence  $\mathbb{E}[X] = \mathbb{P}_2(A) - \mathbb{P}_2(A^c)$  so if  $\mathbb{E}[X] = 0$  we must have  $\mathbb{P}_1(A) = \mathbb{P}_1(A^c)$ . We may write  $\mathbb{E}[X] = k_0 \mathbb{P}_0 + k_1 \mathbb{P}_1 + k_2 \mathbb{P}_2 + k_3 \mathbb{P}_3 + k_4 \mathbb{P}_4 = k_0/16_0 + 4k_1/16 + 6k_2/16 + 4k_3/16 + k_4/16$  which is zero either if  $k_i = (-1)^i$  or if  $k_i = -(-1)^i$ . Hence, there are two possible  $\mathcal{F}_2$ -measurable functions  $X : \Omega \to \{-1, 1\}$  such that  $\mathbb{E}[X] = 0$ .

3) Using the same notation as in 2), we conclude that, for any given integer  $n > 0, A_0, A_1, \ldots, A_n$  is a partition of  $\Omega$  with  $\sigma$ -algebra  $\mathcal{F}_n$  generated by this partition, and with  $\mathbb{P}_n(A_i) = \binom{n}{i}(1/2)^i$  for  $i = 0, 1, \ldots, n$ . And  $\mathbb{E}[X] = \sum_{i=0}^n k_i \binom{n}{i}(1/2)^i$  for  $k_i = \pm 1$ , where it should be noted that  $\sum_{i=0}^n \binom{n}{i}(1/2)^i = 1$ . For any function  $X : \Omega \to \{-1, 1\}$  that is measurable with respect to  $\mathcal{F}_n$  there is a set  $A \in \mathcal{F}_n$  such that

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ -1 & \text{if } \omega \in A^c. \end{cases}$$

Hence  $\mathbb{E}[X] = \mathbb{P}_n(A) - \mathbb{P}_n(A^c)$  so if  $\mathbb{E}[X] = 0$  we must have  $\mathbb{P}_n(A) = \mathbb{P}_n(A^c) = 1/2$ .

For even n, we have

$$\sum_{i=odd} {\binom{n}{i}} (1/2)^i = 1/2$$
$$\sum_{i=even} {\binom{n}{i}} (1/2)^i = 1/2$$

so  $\mathbb{E}[X] = 0$  if  $k_i = (-1)^i$  or  $k_i = -(-1)^i$ . For odd n, we have

$$\sum_{i=0}^{(n-1)/2} {\binom{n}{i}} (1/2)^i = 1/2$$
$$\sum_{i=(n-1)/2+1}^{n} {\binom{n}{i}} (1/2)^i = 1/2$$

so  $\mathbb{E}[X] = 0$  if  $k_{n-i} = -k_i$  for  $i = 0, 1, \dots, (n-1)/2$  and there are  $2^{(n-1)/2}$  ways of combining these. Hence, there are  $2^{(n-1)/2}$  possible  $\mathcal{F}_n$ -measurable functions  $X : \Omega \to \{-1, 1\}$  such that  $\mathbb{E}[X] = 0$ .  $\Box$ 

## 4.13

Consider the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  where  $\Omega = [0, 1]$ ,  $\mathcal{F}$  is the Borel  $\sigma$ algebra on  $\Omega$  and  $\mathbb{P}$  is the uniform probability measure on  $(\Omega, \mathcal{F})$ . Show that the two random variables  $X(\omega) = \omega$  and  $Y(\omega) = 2|\omega - 1/2|$  have the same distribution, but that  $\mathbb{P}(X = Y) = 0$ .

*Proof.* (For definition of distribution function see definition 3.4) Since X is uniform, the distribution function is

$$\mathbb{P}(X \le x) = 0 \quad \text{for} \quad x \in (-\infty, 0]$$
  
$$\mathbb{P}(X \le x) = x \quad \text{for} \quad x \in [0, 1]$$
  
$$\mathbb{P}(X \le x) = 1 \quad \text{for} \quad x \in [1, \infty)$$

so for Y we get, with  $x \in [0, 1]$ ,

$$\mathbb{P}(Y \le x) = \mathbb{P}(2|X - 1/2| \le x) = \mathbb{P}(|2X - 1| \le x) = \{\text{by symmetry}\}\$$
  
=  $2\mathbb{P}(0 \le 2X - 1 \le x) = 2\mathbb{P}(1/2 \le X \le (x + 1)/2)$   
=  $2((x + 1)/2 - 1/2) = x.$ 

Hence  $\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x) = x$  for  $x \in [0, 1]$ . But since

$$\begin{split} \mathbb{P}(X=Y) &= \mathbb{P}(X=2|X-1/2|) \\ &= \mathbb{P}(\{\omega \in \Omega : X=2X-1/2\} \cup \{\omega \in \Omega : X=-2X+1/2\}) \\ &= \mathbb{P}(\{\omega \in \Omega : X=1\} \cup \{\omega \in \Omega : X=3\}) = \mathbb{P}(\emptyset) = 0, \end{split}$$

we conclude that P(X = Y) = 0.

### 4.14

Show that the smallest  $\sigma$ -algebra containing a set A is that intersection of all  $\sigma$ -algebras containing A. Also, show by counter example that the union of two  $\sigma$ -algebras is not necessarily a  $\sigma$ -algebra.

*Proof.* Let  $\mathcal{G}_A$  be the set of all  $\sigma$ -algebras containing A, we want to show that  $\cap_{\mathcal{G}\in\mathcal{G}_A}\mathcal{G} = \{\omega \in \Omega : \omega \in \mathcal{G} \text{ for every } \mathcal{G} \in \mathcal{G}_A\} = \mathcal{F}$  is a  $\sigma$ -algebra; if it is, then it is the smallest  $\sigma$ -algebra since otherwise, there would be a  $\sigma$ -algebra containing A that was smaller than  $\mathcal{F}$ , this would be a contradiction since then this  $\sigma$ -algebra would be included in  $\mathcal{G}$  and therefore  $\mathcal{F}$  would not be minimal. To check that  $\mathcal{F}$  is a  $\sigma$ -algebra, we simply use Definition 1.3.

- 1.  $\emptyset \in \mathcal{G} \ \forall \mathcal{G} \in \mathcal{G}_A$  hence  $\emptyset \in \mathcal{F}$ .
- 2. If  $\omega \in \mathcal{G} \ \forall \mathcal{G} \in \mathcal{G}_A$  then  $\omega^c \in \mathcal{G} \ \forall \mathcal{G} \in \mathcal{G}_A$  hence  $\omega^c \in \mathcal{F}$ .
- 3. If  $\omega_1, \omega_2, \ldots \in \mathcal{G} \ \forall \mathcal{G} \in \mathcal{G}_A$  then  $\bigcup_{i=1}^{\infty} \omega_i \in \mathcal{G} \ \forall \mathcal{G} \in \mathcal{G}_A$  hence  $\omega_1, \omega_2, \ldots \in \mathcal{F}$ .

so  $\mathcal{F}$  is a  $\sigma$ -algebra .

To show that the union of two  $\sigma$ -algebras is not necessarily a  $\sigma$ -algebra , take  $\mathcal{F}_1 = \{\emptyset, \Omega, A, A^c\}$  and  $\mathcal{F}_2 = \{\emptyset, \Omega, B, B^c\}$  where  $A \subset B$ . Then  $\mathcal{F}_1 \cup \mathcal{F}_2 = \{\omega \in \Omega : \omega \in \mathcal{F}_1 \text{ or } \mathcal{F}_2\} = \{\emptyset, \Omega, A, A^c, B, B^c\}$  is not a  $\sigma$ -algebra since, e.g.  $B^c \cup A \notin \mathcal{F}_1 \cup \mathcal{F}_2$ .

## 4.21

given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a random variable X. Let  $\mathcal{A}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ , and consider  $\hat{X} = \mathbb{E}[X|\mathcal{A}]$ .

- 1. Show that  $\mathbb{E}[(X \hat{X})Y] = 0$  if Y is an  $\mathcal{A}$ -measurable random variable.
- 2. Show that the  $\mathcal{A}$ -measurable random variable Y that minimize  $\mathbb{E}[(X-Y)^2]$  is  $Y = \mathbb{E}[X|\mathcal{A}]$ .

Proof. 1) Using equality 4 followed by equality 2 in Proposition 4.8 we get

$$\mathbb{E}[(X - \hat{X})Y] = \mathbb{E}[\mathbb{E}[(X - \hat{X})Y|\mathcal{A}]] = \mathbb{E}[\mathbb{E}[(X - \hat{X})|\mathcal{A}]Y]$$
$$= \mathbb{E}[(\hat{X} - \hat{X})Y] = 0.$$

2) Using equality 3 followed by equality 2 in Proposition 4.8 we get

$$\mathbb{E}[(X-Y)^2] = \mathbb{E}[X^2] - 2\mathbb{E}[XY] + \mathbb{E}[Y^2]$$
  

$$= \mathbb{E}[X^2] - 2\mathbb{E}[\mathbb{E}[XY|\mathcal{A}]] + \mathbb{E}[Y^2]$$
  

$$= \mathbb{E}[X^2] - 2\mathbb{E}[Y\mathbb{E}[X|\mathcal{A}]] + \mathbb{E}[Y^2]$$
  

$$= \mathbb{E}[X^2] - \mathbb{E}[\mathbb{E}[X|\mathcal{A}]^2] + \mathbb{E}[\mathbb{E}[X|\mathcal{A}]^2] - 2\mathbb{E}[Y\mathbb{E}[X|\mathcal{A}]] + \mathbb{E}[Y^2]$$
  

$$= \mathbb{E}[X^2] - \mathbb{E}[\mathbb{E}[X|\mathcal{A}]^2] + \underbrace{\mathbb{E}[(Y-\mathbb{E}[X|\mathcal{A}])^2]}_{\geq 0},$$

hence  $\mathbb{E}[(X - Y)^2]$  is minimized when  $Y = \mathbb{E}[X|\mathcal{A}]$ .

## 4.22

Let X and Y be two integrable random variables defined on the same probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $\mathcal{A}$  be a sub- $\sigma$ -algebra such that X is  $\mathcal{A}$ -measurable.

- 1. Show that  $\mathbb{E}[Y|\mathcal{A}] = X$  implies that  $\mathbb{E}[Y|X] = X$ .
- 2. Show by counter example that  $\mathbb{E}[Y|X] = X$  does not necessarily imply that  $\mathbb{E}[Y|\mathcal{A}] = X$ .

*Proof.* (For definition of conditional expectation see Definition 4.6, for properties of the conditional expectation, see Proposition 4.8.)

1) Since X is  $\mathcal{A}$ -measurable,  $\sigma(X) \subseteq \mathcal{A}$  hence, using equality 4 in Proposition 4.8, we get

$$\mathbb{E}[Y|X] \stackrel{\Delta}{=} \mathbb{E}[Y|\underbrace{\sigma(X)}_{\subseteq \mathcal{A}}] = \mathbb{E}[\underbrace{\mathbb{E}[Y|\mathcal{A}]}_{=X}|X] = \mathbb{E}[X|X] = X.$$

2) Let X and Z be independent and integrable random variables and assume that  $\mathbb{E}[Z] = 0$  and define Y = X + Z. Let  $\mathcal{A} = \sigma(X, Z)$ , then

$$\mathbb{E}[Y|\sigma(X)] = \mathbb{E}[X + Z|\sigma(X)] = X + 0 = X$$

while

$$\mathbb{E}[Y|\mathcal{A}] = \mathbb{E}[X + Z|\sigma(X, Z)] = X + Z \neq X.$$

# Chapters 5 to 7

## 7.1

let  $X_i$  be a sequence of independent random variables with  $\mathbb{E}[X_i] = 0$  and  $V(X_i) = \mathbb{E}[(X_i - \mathbb{E}[X_i])^2] = \sigma_i^2$ . Show that the sequence

$$S_n = \sum_{i=1}^n (X_i^2 - \sigma_i^2)$$

is a martingale with respect to  $\mathcal{F}$ , the filtration generated by the sequence  $\{X_i\}$ .

*Proof.* (For definition of martingale see Definition 5.2) We first check the integrability os  $S_n$ .

$$\mathbb{E}[|S_n|] = \mathbb{E}[|\sum_{i=1}^n X_i^2 - \sigma_i^2|] \le \mathbb{E}[\sum_{i=1}^n |X_i^2 - \sigma_i^2|] \le \sum_{i=1}^n \mathbb{E}[|X_i^2 - \sigma_i^2|] \le \sum_{i=1}^n \mathbb{E}[X_i^2] + \mathbb{E}[\sigma_i^2] = \sum_{i=1}^n 2\sigma_i^2 < \infty$$

hence  $S_n$  is integrable.

To check that  $S_n$  is  $\mathcal{F}_n$  -measurable, just observe that since  $X_i$  for  $i = 1, 2, \ldots, n$  are  $\mathcal{F}_n$  measurable, the sum of them is as well.

What remains to prove is the matringale property.

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_n + X_{n+1}^2 - \sigma_{n+1}^2|\mathcal{F}_n] = \mathbb{E}[S_n|\mathcal{F}_n] + \mathbb{E}[X_{n+1}^2|\mathcal{F}_n] - \sigma_{n+1}^2$$
  
=  $S_n + \mathbb{E}[X_{n+1}^2] - \sigma_{n+1}^2 = \mathbb{E}[S_n|\mathcal{F}_n] + \sigma_{n+1}^2 - \sigma_{n+1}^2$   
=  $S_n$ .

Hence  $S_n$  is an  $\underline{\mathcal{F}}$ -martingale.

## 7.4

Let  $X_i$  be IID with  $X_i \sim N(0,1)$  for each *i* and put  $Y_n = \sum_{i=1}^n X_i$ . Show that

$$S_n = exp\{\alpha Y_n - n\alpha^2/2\}$$

is an  $\mathcal{F}_n$ -martingale for every  $\alpha \in \mathbb{R}$ .

*Proof.* (For definition of martingale see Definition 5.2) We first check the integrability os  $S_n$ . Since  $S_n \geq 0$  and by knowing that  $\mathbb{E}[e^{cX}] = e^{c^2/2}$  for  $X \sim N(0, 1)$  (moment generating function), we get

$$\mathbb{E}[|S_n|] = \mathbb{E}[S_n] = \mathbb{E}[\exp\{\alpha \sum_{i=1}^n X_i - n\alpha^2/2\}] = e^{-n\alpha^2/2} \prod_{i=1}^n \mathbb{E}[e^{\alpha X_i}] = 1 < \infty.$$

hence  $S_n$  is integrable.

To check that  $S_n$  is  $\mathcal{F}_n$ -measurable, observe that since  $X_i$  for  $i = 1, 2, \ldots, n$  are  $\mathcal{F}_n$  measurable, and the exponential of the sum is continuous, so  $S_n$  is measurable as well.

What remains to prove is the matringale property.

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}[\exp\{\alpha \sum_{i=1}^n X_i - n\alpha^2/2\} |\mathcal{F}_n] = \mathbb{E}[S_n \exp\{\alpha X_{n+1} - \alpha^2/2\} |\mathcal{F}_n]$$
$$= S_n \underbrace{\mathbb{E}[\exp\{\alpha X_{n+1} - \alpha^2/2\} |\mathcal{F}_n]}_{=1 \text{ (as in 7.1)}} = S_n.$$

Hence  $S_n$  is an  $\underline{\mathcal{F}}$ -martingale.

### 7.13

Let  $X_i$  be a sequence of bounded random variables such that

$$S_n = \sum_{i=1}^n X_i$$

is an  $\mathcal{F}$ -martingale. Show that  $Cov(X_i, X_j) = 0$  for  $i \neq j$ .

*Proof.* By Proposition 5.4 we get that  $\mathbb{E}[X_i] = 0$  for i > 1, hence for  $1 \le n \le n + m$  we have

$$\operatorname{Cov}(X_n, X_{n+m}) = \mathbb{E}[X_n X_{n+m}] - \mathbb{E}[X_n] \underbrace{\mathbb{E}[X_{n+m}]}_{=0} = \mathbb{E}[\mathbb{E}[X_n X_{n+m} | \mathcal{F}_n]]$$
$$= \mathbb{E}[X_n \mathbb{E}[X_{n+m} | \mathcal{F}_n]] = \mathbb{E}[X_n \mathbb{E}[S_{n+m} - S_{n+m-1} | \mathcal{F}_n]] = 0$$

where the last equality stems from the fact that  $S_n$  is an  $\underline{\mathcal{F}}$ -martingale. Hence the sequence  $X_i$  are mutually uncorrelated.

# Chapter 8

## 8.1

Let  $\{M_n\}$  and  $\{N_n\}$  be square integrable  $\underline{\mathcal{F}}$ -martingales. Show that

$$\mathbb{E}[M_{n+1}N_{n+1}|\mathcal{F}_n] - M_n N_n = \langle M, N \rangle_{n+1} - \langle M, N \rangle_n \tag{1}$$

*Proof.* (For definition of square integrability see Definition 8.1, for definition of quadratic variatio and covariation see page 54) The right hand side of equality (1) yields

$$\langle M, N \rangle_{n+1} - \langle M, N \rangle_n = = \sum_{i=0}^n \mathbb{E}[(M_{i+1} - M_i)(N_{i+1} - N_i)|\mathcal{F}_i] - \sum_{i=0}^{n-1} \mathbb{E}[(M_{i+1} - M_i)(N_{i+1} - N_i)|\mathcal{F}_i] = \mathbb{E}[(M_{n+1} - M_n)(N_{n+1} - N_n)|\mathcal{F}_n] = \mathbb{E}[M_{n+1}N_{n+1} - M_{n+1}N_n - M_nN_{n+1} + M_nN_n|\mathcal{F}_n].$$

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Using the martingale property of the two processes  $\{M_n\}$  and  $\{N_n\}$ , and the measurability of  $M_n$  and  $N_n$  with respect to  $\mathcal{F}_n$ , we get

$$\begin{split} & \mathbb{E}[M_{n+1}N_{n+1} - M_{n+1}N_n - M_nN_{n+1} + M_nN_n|\mathcal{F}_n] \\ &= \mathbb{E}[M_{n+1}N_{n+1}|\mathcal{F}_n] - \mathbb{E}[M_{n+1}N_n|\mathcal{F}_n] - \mathbb{E}[M_nN_{n+1}|\mathcal{F}_n] + \mathbb{E}[M_nN_n|\mathcal{F}_n] \\ &= \mathbb{E}[M_{n+1}N_{n+1}|\mathcal{F}_n] - N_n\mathbb{E}[M_{n+1}|\mathcal{F}_n] - M_n\mathbb{E}[N_{n+1}|\mathcal{F}_n] + M_nN_n \\ &= \mathbb{E}[M_{n+1}N_{n+1}|\mathcal{F}_n] - N_nM_n, \end{split}$$

and the proof is done.

## 8.2

Let  $\{M_n\}$  and  $\{N_n\}$  be square integrable  $\underline{\mathcal{F}}$ -martingales.

1. Let  $\alpha$  and  $\beta$  be real numbers. verify that, for every integer  $n \ge 0$ ,

$$\langle \alpha M + \beta N \rangle_n = \alpha^2 \langle M \rangle_n + 2\alpha \beta \langle M, N \rangle_n + \beta^2 \langle N \rangle_n.$$

2. Derive the *Cauchy-Schwarz* inequality

$$|\langle M,N\rangle_n| \leq \sqrt{\langle M\rangle_n} \sqrt{\langle N\rangle_n}, \ n \geq 0.$$

*Proof.* (For definition of square integrability see Definition 8.1, for definition of quadratic variatio and covariation see page 54) 1) By Definition 8.3 we get

$$\begin{split} \langle \alpha M + \beta N \rangle_n &= \sum_{i=0}^{n-1} \mathbb{E}[(\alpha M_{i+1} + \beta N_{i+1} - \alpha M_i - \beta N_i)^2 |\mathcal{F}_i] \\ &= \sum_{i=0}^{n-1} \mathbb{E}[((\alpha M_{i+1} - \alpha M_i) + (\beta N_{i+1} - \beta N_i))^2 |\mathcal{F}_i] \\ &= \sum_{i=0}^{n-1} \mathbb{E}[\alpha^2 (M_{i+1} - M_i)^2 + 2\alpha\beta (M_{i+1} - M_i)(N_{i+1} - N_i) + \beta^2 (N_{i+1} - N_i)^2 |\mathcal{F}_i] \\ &= \alpha^2 \langle M \rangle_n + 2\alpha\beta \langle M, N \rangle_n + \beta^2 \langle N \rangle_n, \end{split}$$

which is what we wherev set out to prove.

2) It is easily seen that the quadratic variation is alway positive and by using this observation, combined with the result from the first part of this exercise, we get, for any  $\lambda \in \mathbb{R}$ ,

$$0 \le \langle M - \lambda N \rangle_n = \langle M \rangle_n - 2\lambda \langle M, N \rangle_n + \lambda^2 \langle N \rangle_n.$$

Let  $\lambda = \langle M, N \rangle_n / \langle N \rangle_n$ ,

$$0 \le \langle M \rangle_n - 2\lambda \langle M, N \rangle_n + \lambda^2 \langle N \rangle_n$$
  
=  $\langle M \rangle_n - 2\langle M, N \rangle_n^2 / \langle N \rangle_n + \langle M, N \rangle_n^2 / \langle N \rangle_n = \langle M \rangle_n - \langle M, N \rangle_n^2 / \langle N \rangle_n$ 

hence

$$\langle M, N \rangle_n^2 / \langle N \rangle_n \le \langle M \rangle_n$$
$$\iff \ \langle M, N \rangle_n \le \sqrt{\langle M \rangle_n} \sqrt{\langle N \rangle_n}$$

and the proof is done.

## 8.3

Let  $\{M_n\}$  and  $\{N_n\}$  be square integrable  $\underline{\mathcal{F}}$ -martingales. Check the following parallellogram equality,

$$\langle M, N \rangle_n = \frac{1}{4} (\langle M + N \rangle_n - \langle M - N \rangle_n).$$

Proof. Using the result from part 1) of problem 8.2 we get

$$\langle M+N\rangle_n - \langle M-N\rangle_n = \langle M\rangle_n + 2\langle M,N\rangle_n + \langle N\rangle_n - \langle M\rangle_n + 2\langle M,N\rangle_n - \langle N\rangle_n = 4\langle M,N\rangle_n,$$

hence  $\langle M, N \rangle_n = \frac{1}{4} (\langle M + N \rangle_n - \langle M - N \rangle_n).$ 

### 

## Chapter 9

### 9.2

Let  $\{M_n\}$  and  $\{N_n\}$  be two square integrable  $\mathcal{F}$ -martingales and let  $\varphi$  and  $\psi$  be bounded  $\mathcal{F}$ -adapted processes. Derive the Cauchy-Schwarz inequality

$$|\langle I_M(\varphi), I_N(\psi) \rangle_n| \le \sqrt{\langle I_M(\varphi) \rangle_n} \sqrt{\langle I_N(\psi) \rangle_n}, \ n \ge 0.$$

*Proof.* By Proposition 9.3 we have that both  $I_M(\varphi)$  and  $I_N(\psi)$  are square integrable  $\mathcal{F}$ -martingales, so the proof is identical to the one given in part 2) of problem 8.2.

## 9.3

In this problem we look at a simple market with only two assets; a bond and a stock. The bond price is modelled according to

$$\begin{cases} B_n = (1+r)B_{n-1} & \text{for } n = 1, 2, \dots, N \\ B_0 = 1 \end{cases}$$

where r > -1 is the constant rate of return for the bond. The stock price is asumed to be stochastic, with dynamics

$$\begin{cases} S_n = (1+R_n)S_{n-1} & \text{for } n = 1, 2, \dots, N \\ S_0 = s \end{cases}$$

where s > 0 and  $\{R_n\}$  is a sequence of IID random variables on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Furthermore, let  $\{\mathcal{F}_n\}$  be the filtration given by  $\mathcal{F}_n = \sigma(R_1, \ldots, R_n)$   $n = 1, \ldots, N$ 

a) When is  $S_n/B_n$  a martingale with respect to the filtration  $\{\mathcal{F}_n\}$ ?

We now look at portfolios consisting of the bond and the stock. For every n = 0, 1, 2, ..., N let  $x_n$  and  $y_n$  be the number of stocks and bonds respectively bought at time n and held over the period [n, n + 1). furthermore, let

$$V_n = x_n S_n + y_n B_n$$

by the value of the portfolio over [n, n + 1), and let  $V_0$  be our initial wealth. The rebalancing of the portfolio is done in the following way.

At every time n we observe the value of our old portfolio, composed at time n-1, which at time n is  $x_{n-1}S_n + y_{n-1}B_n$ . We are allowed to only use this amount to to rebalance the portfolio at time n, i.e. we are not allowed to withdraw or add any money to the portfolio. A portfolio with this restriction is called a *self* - *financingportfolio*. Formally we define a self-financing portfolio as a pair  $\{x_n, y_n\}$  of  $\{\mathcal{F}_n\}$ -adapted processes such that

$$x_{n-1}S_n + y_{n-1}B_n = x_nS_n + y_nB_n, \ n = 1, \dots, N.$$

b) Show that if  $S_n/B_n$  is a martingale with respect to the filtration  $\{\mathcal{F}_n\}$ , then so is  $V_n/B_n$ , where  $V_n$  is the portfolio value of any self-financing portfolio.

Finally we look at a type of self-financing portfolios called *arbitrage strategies*. A portfolio  $\{x_n, y_n\}$  is called an arbitrage if we have

$$V_0 = 0$$
$$\mathbb{P}(V_N \ge 0) = 1$$
$$\mathbb{P}(V_N > 0) > 0$$

for the value process of the portfolio. The idea formalized in an arbitrage portfolio is that with an initial wealth of 0 we get a non-negative portfolio value at time N with probability one, i.e. your are certain to make money on your strategy. We say that a model is *arbitrage free* if the model permits arbitrage portfolios.

c) Show that if  $S_n/B_n$  is a martingale then every self-financing portfolio is arbitrage free.

Let  $Q_n$  be a square integrable martingale with respect to the filtration  $\{\mathcal{F}_n\}$  such that  $Q_n > 0$  a.s. and  $Q_0 = 1$  a.s..

d) Show that even if  $S_n/B_n$  is not a martingale with respect to the filtration  $\{\mathcal{F}_n\}$ , finding a process  $Q_n$  as defined above such that  $S_nQ_n/B_n$ , will give that  $V_nQ_n/B_n$  is a martingale with respect to the filtration  $\{\mathcal{F}_n\}$ , and furthermore, that  $V_n$  is arbitrage free.

Even though the multiplication of the positive martingale  $Q_n$  might seem unimportant, we will later in the course see that this is in fact a very special action which gives us the ability to *change measure*. In financial applications, this is important since the portfolio pricing theory say that a portfolio should be priced under a *risk neutral measure*, a measure where all portfolios, divided by the bank process  $B_n$  should be a martingale. The reason for this is that the theory is based on a no-arbitrage assumption, which hold if  $S_n/B_n$  or  $S_nQ_n/B_n$ is a martingale as proven in this exercise. So the existence of  $Q_n$  guarantees that  $V_n$  is arbitrage free, and using a change of measure closely related to  $Q_n$ we may price any portfolio  $V_n$  consisting of  $S_n$  and  $B_n$  in a consistent way.

*Proof.* a) Use Definition 5.2 to conclude that  $S_n/B_n$  is an  $\{\mathcal{F}_n\}$ -martingale if the process is integrable, measurable and have the martingale property, i.e. that

 $\mathbb{E}[S_{n+1}/B_{n+1}|\mathcal{F}_n] = S_n/B_n$ . Since  $S_n/B_n \ge 0$  for every  $n, B_n$  is deterministic and the  $R_n$ 's are IID we get

$$\mathbb{E}[|S_n/B_n|] = \mathbb{E}[S_n/B_n] = \frac{\prod_{i=1}^n \mathbb{E}[1+R_n]}{\prod_{i=1}^n (1+r)} = \prod_{i=1}^n \frac{\mathbb{E}[1+R_n]}{1+r},$$
(2)

hence  $S_n/B_n$  is integrable if  $R_n$  is. Since  $\mathcal{F}_n = \sigma(R_1, \ldots, R_n)$ , and the produkt is a continuous mapping,  $S_n/B_n$  is  $\mathcal{F}_n$ -measurable. To check the martingale property, just add a conditioning to (2),

$$\mathbb{E}[S_{n+1}/B_{n+1}|\mathcal{F}_n] = \frac{S_n \mathbb{E}[1+R_{n+1}|\mathcal{F}_n]}{B_n(1+r)} = \frac{S_n}{B_n} \frac{\mathbb{E}[1+R_n|\mathcal{F}_n]}{1+r} = \frac{S_n}{B_n} \frac{1+\mathbb{E}[R_n|\mathcal{F}_n]}{1+r}$$

To get the martingale property  $\mathbb{E}[S_{n+1}/B_{n+1}|\mathcal{F}_n] = S_n/B_n$  we must have

$$\frac{1 + \mathbb{E}[R_n | \mathcal{F}_n]}{1 + r} = 1,$$

or equivalently that  $\mathbb{E}[R_{n+1}|\mathcal{F}_n] = r$ . Hence  $S_n/B_n$  is an  $\{\mathcal{F}_n\}$ -martingale if  $\mathbb{E}[R_{n+1}|\mathcal{F}_n] = r$ .

b) Since the definition of a self-financing portfolio is that

$$x_{n-1}S_n + y_{n-1}B_n = x_nS_n + y_nB_n, \ n = 1, \dots, N$$

we get, by the definition of  $V_n$ ,

$$\mathbb{E}\begin{bmatrix}\frac{V_{n+1}}{B_{n+1}}|\mathcal{F}_n] = \mathbb{E}\begin{bmatrix}\frac{x_{n+1}S_{n+1} + y_{n+1}B_{n+1}}{B_{n+1}}|\mathcal{F}_n] = \mathbb{E}\begin{bmatrix}\frac{x_nS_{n+1} + y_nB_{n+1}}{B_{n+1}}|\mathcal{F}_n\end{bmatrix}$$
$$= x_n\mathbb{E}\begin{bmatrix}\frac{S_{n+1}}{B_{n+1}}|\mathcal{F}_n] + y_n\frac{B_{n+1}}{B_{n+1}}$$

since  $\{x_n, y_n\}$  are  $\mathcal{F}_n$ -measurable. Under the assuption that  $S_n/B_n$  is an  $\mathcal{F}_n$ -martingale we get

$$\mathbb{E}[\frac{V_{n+1}}{B_{n+1}}|\mathcal{F}_n] = \mathbb{E}[\frac{x_n S_{n+1} + y_n B_{n+1}}{B_{n+1}}|\mathcal{F}_n] = \frac{x_n S_n + y_n B_n}{B_n} = \frac{V_n}{B_n},$$

so  $V_n/B_n$  is an  $\mathcal{F}_n$ -martingale if  $S_n/B_n$  is.

c) From b) we have that any self-financing portfolio  $V_n = x_n S_n + y_n B_n$  is such that  $V_n/B_n$  is a martingale if  $S_n/B_n$  is. To check that any self-financing portfolio  $V_n$  is arbitrage free, we must have  $V_0 = x_0 S_0 + y_0 B_0 = 0$ . Let  $S_n/B_n$ be a martingale, then by Proposition 5.4 a) we have

$$\mathbb{E}[V_{n+1}/B_{n+1}] = V_0/B_0 = V_0 = 0.$$

Assume that  $P(V_n \ge 0) = 1$  and  $P(V_n > 0) > 0$ . Since  $\mathbb{E}[V_n/B_n] = 0$  and  $B_n < \infty$  for any n we get

$$\mathbb{E}\begin{bmatrix}\frac{V_n}{B_n}\end{bmatrix} = \underbrace{\mathbb{E}\begin{bmatrix}\frac{V_n}{B_n}I_{\{V_n=0\}}\end{bmatrix}}_{=0} + \underbrace{\mathbb{E}\begin{bmatrix}\frac{V_n}{B_n}I_{\{V_n>0\}}\end{bmatrix}}_{>0} > 0,$$

where  $I_{\{\cdot\}}$  is the indicator function. This is a contradiction to  $\mathbb{E}[V_n/B_n] = 0$ , hence there are no arbitrage strategies  $V_n$ .

d) Following the same lines as in b) we get that if  $S_n Q_n / B_n$  is a martingale with respect to the filtration  $\{\mathcal{F}_n\}$  and  $V_n$  is self financing,

$$\mathbb{E}\left[\frac{V_{n+1}Q_{n+1}}{B_{n+1}}|\mathcal{F}_n\right] = \mathbb{E}\left[\frac{x_n S_{n+1}Q_{n+1} + y_n B_{n+1}Q_{n+1}}{B_{n+1}}|\mathcal{F}_n\right] = \frac{x_n S_n Q_n + y_n B_n Q_n}{B_n} = \frac{V_n Q_n}{B_n}$$

so  $V_n Q_n / B_n$  is an  $\mathcal{F}_n$ -martingale if  $S_n Q_n / B_n$  is. And following the same lines as the proof of c),

$$\mathbb{E}\left[\frac{V_{n+1}Q_{n+1}}{B_{n+1}}\right] = \frac{V_0Q_0}{B_0} = V_0 = 0.$$

Assume that  $P(V_n \ge 0) = 1$  and  $P(V_n > 0) > 0$ . Since  $\mathbb{E}[V_n Q_n / B_n] = 0$  and  $Q_n > 0$ ,  $B_n < \infty$  for any n we get

$$\mathbb{E}[\frac{V_nQ_n}{B_n}] = \underbrace{\mathbb{E}[\frac{V_nQ_n}{B_n}I_{\{V_n=0\}}]}_{=0} + \underbrace{\mathbb{E}[\frac{V_nQ_n}{B_n}I_{\{V_n>0\}}]}_{>0} > 0,$$

where  $I_{\{\cdot\}}$  is the indicator function. This is a contradiction to  $\mathbb{E}[V_n Q_n / B_n] = 0$ , hence there are no arbitrage strategies  $V_n$ .

#### 9.4

A coin is tossed N times, where the number N is known in advance. 1 unit invested in a coin toss gives the net profit of 1 unit with probability  $p \in (1/2, 1]$  and the net profit of -1 with probability 1 - p. If we let  $X_n$  n = 1, 2, ..., N be the net profit per unit invested in the  $n^{\text{th}}$  coin toss, then,

$$\mathbb{P}(X_n = 1) = p \text{ and } \mathbb{P}(X_n = -1) = 1 - p,$$

and the  $X_n$ 's are independent of each other. Let  $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$  and let  $S_n, n = 1, 2, \ldots, N$  ne the wealth of the investor at time n. Assume further that the initial wealth  $S_0$  is a given constant. Any non-nergative amount  $C_n$  can be invested in coin toss  $n + 1, n = 1, \ldots, n - 1$ , but we assume that borrowing money is not allowed so  $C_n \in [0, S_n]$ . Thus we have

$$S_{n+1} = S_n + C_n X_{n+1}, n = 1, \dots, N-1 \text{ and } C_n \in [0, S_n]$$

Finally assume that the objective of the investor is to maximize the expected rate of return  $\mathbb{E}[(1/N)\log(S_N/S_0)]$ .

a) Show that  $S_n$  is a submartingale with respect to the filtration  $\{\mathcal{F}\}$ .

b) Show that whatever strategy  $C_n$  the investor use in the investment game,  $L_n = log(S_n) - n\alpha$  where  $\alpha = p \log(p) + (1-p) \log(1-p) + \log(2)$  is a supermartingale with respect to the filtration  $\{\mathcal{F}_n\}$ .

*Hint:* At some point you need to study the function

$$g(x) = p \log(1+x) + (1-p) \log(1-x)$$
 for  $x \in [0,1]$  and  $p \in (1/2,1)$ .

c) Show that the fact that  $\log(S_n) - n\alpha$  is a supermartingale implies that  $\mathbb{E}[[\log(S_N/S_0)] \le N\alpha.$ 

d) Show that if  $C_n = S_n(2p-1)$ ,  $L_n$  is an  $\{\mathcal{F}\}$ -martingale.

Proof. (For definition of submartingale and supermartingale see the text following Definition 5.2) a) To show that  $S_n$  is a submartingale w.r.t.  $\{\mathcal{F}\}$  we want to show that  $\mathbb{E}[S_{n+1}|\mathcal{F}_n] \geq S_n$ ,

$$\mathbb{E}[S_{n+1}|\mathcal{F}_n] = \mathbb{E}[S_n + C_n X_{n+1}|\mathcal{F}_n] = \{C_n \text{ and } S_n \text{ are } \mathcal{F}_n\text{-measurable}\}$$
$$= S_n + C_n \mathbb{E}[X_{n+1}|\mathcal{F}_n] = \{X_{n+1} \text{ independent of } \mathcal{F}_n\}$$
$$= S_n + C_n \mathbb{E}[X_{n+1}] = S_n + \underbrace{C_n}_{\geq 0} \underbrace{(1 \cdot p - 1 \cdot (1 - p))}_{>0} \ge S_n,$$

hence  $S_n$  is a submartingale w.r.t.  $\{\mathcal{F}_n\}$ .

b) We now want to show that 
$$\mathbb{E}[L_{n+1}|\mathcal{F}_n] \leq L_n$$
,  
 $\mathbb{E}[L_{n+1}|\mathcal{F}_n] = \mathbb{E}[\log(S_{n+1}) - (n+1)\alpha|\mathcal{F}_n] = \mathbb{E}[\log(S_n + C_n X_{n+1})|\mathcal{F}_n]$   
 $- (n+1)\alpha = \mathbb{E}[\log(S_n(1 + C_n X_{n+1}/S_n))|\mathcal{F}_n] - (n+1)\alpha$   
 $= \mathbb{E}[\log(S_n) + \log(1 + C_n X_{n+1}/S_n))|\mathcal{F}_n] - (n+1)\alpha$   
 $= \underbrace{\log(S_n) - n\alpha}_{=L_n} + \mathbb{E}[\log(1 + C_n X_{n+1}/S_n))|\mathcal{F}_n] - \alpha$   
 $= L_n + \underbrace{p\log(1 + C_n/S_n) + (1-p)\log(1 - C_n/S_n)}_{=g(C_n/S_n)} - \alpha = L_n + g(C_n/S_n) - \alpha$ 

Since  $g''(x) = -p/(1+x^2) - (1-p)/(1-x^2) < 0$  for  $x \in [0,1)$ , g is concave in that region, and the maximum is  $\hat{x} = 2p-1$  since  $g'(\hat{x}) = p/(1+\hat{x})-(1-p)/(1-\hat{x}) = 0$ so  $g(x) \leq g(\hat{x})$  for all  $x \in [0, 1]$ . Since  $C_n/S_n \in [0, 1]$ ,

$$g(C_n/S_n) \le g(\hat{x}) = g(2p-1) = p\log(p) + (1-p)\log(1-p) + \log 2 = \alpha$$

hence

$$\mathbb{E}[L_{n+1}|\mathcal{F}_n] = L_n + g(C_n/S_n) - \alpha \le L_n + \alpha - \alpha = L_n,$$

so  $L_n$  is a supermartingale w.r.t.  $\{\mathcal{F}_n\}$ .

c) We have just shown that  $L_n = \log(S_n) - n\alpha$  is a supermartingale w.r.t.  $\{\mathcal{F}_n\}$ . Because of this we also have that  $\mathbb{E}[L_n] \leq L_0$  so

$$\mathbb{E}[\log(S_N) - N\alpha] \le \log(S_0) - 0 \cdot \alpha \iff \mathbb{E}[\log(S_N/S_0)] \le N\alpha.$$

d) For  $C_n = S_N(2p-1)$  we get

 $\mathbb{E}[$ 

$$[L_{n+1}|\mathcal{F}_n] = L_n + g(C_n/S_n) - \alpha = L_n + g(2p-1) - \alpha = L_n$$

hence  $L_n$  is a  $\{\mathcal{F}_n\}$ -martingale using the strategy  $C_n = S_n(2p-1)$ .

## 9.5

Assume that  $X_n$  n = 0, 1, 2, ... is that price of a stock at time n and assume that  $X_n$  is a supermartingale with respect to the filtration  $\{\mathcal{F}_n\}$ . This means that if we buy one unit of stock at time n, paying  $X_n$ , the expected price of the stock tomorrow (represented by the time n + 1) given the information  $\mathcal{F}_n$  is lower than today's price. in other words, we expect the price to go down. Investing in the stock does not seem to be a good idea, but is it possible to find a strategy that performs better? The answer is no, and the objective of this exercise is to show that. Let  $C_n$  be a process adapted to  $\{\mathcal{F}_n\}$  with  $0 \leq C_n$  n = 0, 1, 2, ..., representing our investment strategy. We know that the gain of our trading after n days is given by  $I_X(C)_n$ , the sochastic integral of C with respect to X. Now, show that for any supermartingale  $X_n$  and any positive, adapted and bounded process  $C_n$ 

$$\mathbb{E}[I_X(C)_{n+1}|\mathcal{F}_n] \le I_X(C),$$

i.e. that  $I_X(C)_n$  is also a supermartingale.

*Proof.* (For definition of the stochastic integral  $I_X(C)$  see Definition 9.1) We may write the stochastic integral as  $I_X(C)_n = \sum_{i=0}^{n-1} C_i(X_{i+1} - X_i)$  so taking the conditional expectation of the stochastic integral we get

$$\mathbb{E}[I_X(C)_{n+1}|\mathcal{F}_n] = \mathbb{E}[\sum_{i=0}^n C_i(X_{i+1} - X_i)|\mathcal{F}_n]$$
  
=  $\mathbb{E}[\sum_{i=0}^{n-1} C_i(X_{i+1} - X_i) + C_n(X_{n+1} - X_n)|\mathcal{F}_n]$   
=  $\mathbb{E}[I_X(C)_n + C_n(X_{n+1} - X_n)|\mathcal{F}_n] = \{I_X(C)_n \text{ is } \mathcal{F}_n \text{ -measurable}\}$   
=  $I_X(C)_n + \mathbb{E}[C_n(X_{n+1} - X_n)|\mathcal{F}_n] = \{C_n \text{ and } X_n \text{ are } \{\mathcal{F}_n\}\text{-adapted}\}$   
=  $I_X(C)_n + C_n(\mathbb{E}[X_{n+1}|\mathcal{F}_n] - X_n) \leq \{X_n \text{ is supermartingale and } C_n \geq 0\}$   
 $\leq I_X(C)_n + C_n(X_n - X_n) = I_X(C)_n.$ 

Hence  $I_X(C)_n$  is a supermartingale with respect to  $\{\mathcal{F}_n\}$  if  $X_n$  is.

## Chapter 10

### 10.1

Let  $B_n$ , n = 0, 1, 2, ... be a discrete Brownian motion. Show that

$$\frac{B_n}{\langle B \rangle_n} \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty,$$

that is for every  $\epsilon > 0$ 

$$\mathbb{P}\left( \left| \frac{B_n}{\langle B \rangle_n} \right| > \epsilon \right) \to 0 \text{ as } n \to \infty.$$

*Proof.* Recall that for a square integrable random variable X, the Chebyshev's inequality is

$$\mathbb{P}(|X| > \epsilon) \le \frac{\mathbb{E}[X^2]}{\epsilon^2}.$$

Since  $\langle B \rangle_n = n$ , wich is given in the text at page 64 if needed, we get

$$\mathbb{P}\left(\left|\frac{B_n}{\langle B \rangle_n}\right| > \epsilon\right) = \mathbb{P}\left(\left|\frac{B_n}{n}\right| > \epsilon\right) = \mathbb{P}\left(|B_n| > n\epsilon\right) \le \frac{\mathbb{E}[B_n^2]}{(n\epsilon)^2} = \frac{n}{(n\epsilon)^2}$$
$$= \frac{1}{\epsilon^2} \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Hence

$$\frac{B_n}{\langle B\rangle_n} \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

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## 10.2

(Continuation of Exercise 9.3) Assume the value of the bond's rate of return is  $r = e^{\frac{1}{2}\sigma^2} - 1$  for some constant  $\sigma$ . What should be the distribution of the random variable  $(1+R_n)$  in order to model  $\tilde{S}_n \stackrel{\Delta}{=} S_n/B_n$  as a geometric Brownian motion i.e.

$$\tilde{S}_n = s e^{\sigma W_n - \frac{1}{2}n\sigma^2}, \ \tilde{S}_0 = s,$$

where  $W_n$  is a discrete Brownian motion.

*Proof.* from the definition of  $\tilde{S}_n$  we get

$$\tilde{S}_n = \frac{S_n}{B_n} = \frac{(1+R_n)S_{n-1}}{(1+r)B_{n-1}} = \frac{(1+R_n)S_{n-1}}{e^{\frac{1}{2}\sigma^2}B_{n-1}}.$$

We get

$$\frac{\tilde{S}_{n+1}}{\tilde{S}_n} = \frac{(1+R_{n+1})}{e^{\frac{1}{2}\sigma^2}} = \frac{(1+R_{n+1})}{e^{\frac{1}{2}\sigma^2}},$$

so letting  $\tilde{S}_n$  be a geometric Brownian motion, we must have

$$\frac{\tilde{S}_{n+1}}{\tilde{S}_n} = \frac{se^{\sigma W_{n+1} - \frac{1}{2}(n+1)\sigma^2}}{se^{\sigma W_n - \frac{1}{2}n\sigma^2}} = se^{\sigma (W_{n+1} - W_n) - \frac{1}{2}\sigma^2}.$$

Combining the two results we get

$$\frac{\tilde{S}_{n+1}}{\tilde{S}_n} = s e^{\sigma(W_{n+1} - W_n) - \frac{1}{2\sigma^2}} = \frac{(1 + R_{n+1})}{e^{\frac{1}{2}\sigma^2}},$$

which holds if

$$1 + R_{n+1} = se^{\sigma(W_{n+1} - W_n)}.$$

# Chapter 11

## 11.1

Let  $\{M_t\}$  and  $\{N_t\}$  be square integrable  $\underline{\mathcal{F}_t}$ -martingales.

1. Let  $\alpha$  and  $\beta$  be real numbers. verify that, for every  $t \ge 0$ ,

$$\langle \alpha M + \beta N \rangle_t = \alpha^2 \langle M \rangle_t + 2\alpha \beta \langle M, N \rangle_t + \beta^2 \langle N \rangle_t.$$

2. Derive the Cauchy-Schwarz inequality

$$|\langle M, N \rangle_t| \le \sqrt{\langle M \rangle_t} \sqrt{\langle N \rangle_t}, \ t \ge 0.$$

*Proof.* (For definition of square integrability see Definition 11.3, for definition of quadratic variation and covariation see pages 74-75) 1) We use the definition of the covariation process to get

$$\begin{split} \langle \alpha M + \beta N \rangle_t &= \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} (\alpha M_{i+1} + \beta N_{i+1} - \alpha M_i - \beta N_i)^2 \\ &= \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} ((\alpha M_{i+1} - \alpha M_i) + (\beta N_{i+1} - \beta N_i))^2 \\ &= \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} \alpha^2 (M_{i+1} - M_i)^2 + 2\alpha\beta (M_{i+1} - M_i) (N_{i+1} - N_i) + \beta^2 (N_{i+1} - N_i)^2 \\ &= \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} \alpha^2 (M_{i+1} - M_i)^2 + 2\lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} \alpha\beta (M_{i+1} - M_i) (N_{i+1} - N_i) \\ &+ \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} \beta^2 (N_{i+1} - N_i)^2 \\ &\alpha^2 \langle M \rangle_t + 2\alpha\beta \langle M, N \rangle_t + \beta^2 \langle N \rangle_t, \end{split}$$

which is what we wherev set out to prove.

2) Recall the Cauchy-Schwarz inequality for *n*-dimensional euclidean space

$$\sum_{i=1}^{n} a_i b_i \le \sqrt{\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2}.$$

We have

$$\langle M, N \rangle_t = \lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} (M_{i+1} - M_i) (N_{i+1} - N_i)$$
  
 
$$\leq \lim_{\|\Pi\| \to 0} \sqrt{\sum_{i=0}^{n-1} (M_{i+1} - M_i)^2 \sum_{i=0}^{n-1} (N_{i+1} - N_i)^2}$$

and since  $\sqrt{\cdot}$  is continuous the limit may be passed inside the root sign

$$\langle M, N \rangle_t \leq \sqrt{\lim_{\|\Pi\| \to 0} \sum_{i=0}^{n-1} (M_{i+1} - M_i)^2 \sum_{i=0}^{n-1} (N_{i+1} - N_i)^2}$$
$$= \sqrt{\langle M \rangle_t \langle N \rangle_t}$$

and the proof is done.

## 11.2

Let  $\{M_t\}$  and  $\{N_t\}$  be square integrable  $\underline{\mathcal{F}_t}$ -martingales. Check the following parallellogram equality,

$$\langle M, N \rangle_t = \frac{1}{4} (\langle M + N \rangle_t - \langle M - N \rangle_t), \ t \ge 0.$$

*Proof.* Using the result from part 1) of problem 11.1 we get

$$\begin{split} \langle M+N\rangle_t &- \langle M-N\rangle_t \\ &= \langle M\rangle_t + 2\langle M,N\rangle_t + \langle N\rangle_t - \langle M\rangle_t + 2\langle M,N\rangle_t - \langle N\rangle_t = 4\langle M,N\rangle_t, \end{split}$$

hence  $\langle M, N \rangle_t = \frac{1}{4} (\langle M + N \rangle_t - \langle M - N \rangle_t).$ 

## Chapter 12

## 12.1

(The value of a *European Call Option*). In the Black.Scholes model, the price  $S_t$  of a risky asset (i.e. an asset that has no deterministic payoff) at time t is given by the formula

$$S_t = se^{(r - \frac{1}{2}\sigma^2)t + \sigma B_t}$$

where  $B_t$  is a Brownian motion and s is a positive constant representing the initial value of the asset. The value of a European Call option, with maturity time T and strike price K is  $(S_T - K)^+$  at time T. If T > t, compute explicitly

$$\mathbb{E}[(S_T - K)^+ | \mathcal{F}_t].$$

*Proof.* Because of the Markov property of the Brownian motion, any expectation of a function h of the Brownian motion evaluated at time T,  $h(B_T)$ , conditioned on a time t < T is only dependent on the value  $B_t$  and the time to maturity T - t. By Proposition 12.4 we get

$$\mathbb{E}[(S_T - K)^+ | \mathcal{F}_t] = \mathbb{E}[(S_t e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma(B_T - B_t)} - K)^+ | \mathcal{F}_t]$$
  
= {Proposition 12.4} =  $\mathbb{E}^{B_t}[(S_t e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma(B_T - B_t)} - K)^+]$ 

and by the time homogeneity we may write  $B_T - B_t = \sqrt{T - t}X$  where  $X \sim N(0, 1)$ , so

$$\mathbb{E}[(S_T - K)^+ | \mathcal{F}_t] = \mathbb{E}^{B_t}[(S_t e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma(B_T - B_t)} - K)^+]$$
  
=  $\mathbb{E}^{B_t}[(S_t e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma\sqrt{T - t}X} - K)^+]$   
=  $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (S_t e^{(r - \frac{\sigma^2}{2})(T - t) + \sigma\sqrt{T - t}x} - K)^+ e^{-\frac{x^2}{2}} dx.$ 

Since  $(\cdot)^+$  is non-zero only when  $S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma\sqrt{T-t}x} \ge K$  which may be rewritten to get x separated as

$$x \ge \frac{\log\left(\frac{K}{S_t}\right) - (r - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}.$$

Call the right hand side of the inequality  $d_1$ , the integral may be written as

$$\begin{split} \mathbb{E}[(S_{T}-K)^{+}|\mathcal{F}_{t}] &= \frac{1}{\sqrt{2\pi}} \int_{d_{1}}^{\infty} (S_{t}e^{(r-\frac{\sigma^{2}}{2})(T-t)+\sigma\sqrt{T-t}x} - K)e^{-\frac{x^{2}}{2}}dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{d_{1}}^{\infty} S_{t}e^{(r-\frac{\sigma^{2}}{2})(T-t)+\sigma\sqrt{T-t}x}e^{-\frac{x^{2}}{2}}dx - K\underbrace{\frac{1}{\sqrt{2\pi}} \int_{d_{1}}^{\infty} e^{-\frac{x^{2}}{2}}dx}_{\mathbb{P}(X \ge d_{1})} \\ &= \frac{1}{\sqrt{2\pi}} S_{t}e^{r(T-t)} \int_{d_{1}}^{\infty} e^{-\frac{\sigma^{2}}{2}(T-t)+\sigma\sqrt{T-t}x-\frac{x^{2}}{2}}dx - K\underbrace{\frac{1}{\sqrt{2\pi}} \int_{d_{1}}^{\infty} e^{-\frac{x^{2}}{2}}dx}_{\mathbb{P}(X \ge d_{1})} \\ &= \frac{1}{\sqrt{2\pi}} S_{t}e^{r(T-t)} \int_{d_{1}}^{\infty} e^{-\frac{1}{2}(x-\sigma\sqrt{T-t})^{2}}dx - K\mathbb{P}(X \ge d_{1}) \\ &= \{y = x - \sigma\sqrt{T-t}, \ dy = dx\} \\ &= S_{t}e^{r(T-t)}\underbrace{\frac{1}{\sqrt{2\pi}} \int_{d_{1}-\sigma\sqrt{T-t}}^{\infty} e^{-\frac{y^{2}}{2}}dy}_{=\mathbb{P}(X \le d_{1}-\sigma\sqrt{T-t})} - K\mathbb{P}(X \ge d_{1}). \end{split}$$

This is the explicit form of the Call Option price.

### 12.2

Let  $B_t$  be a one dimensional Brownian motion and let  $\mathcal{F}_t$  be the filtration generated by  $B_t$ . Show that

$$\mathbb{E}[B_t^3|\mathcal{F}_s] = B_s^3 + 3(t-s)B_s.$$

*Proof.* We start by separating the process into a part that is measurable with respect to  $\mathcal{F}_s$  and one that is independent of  $\mathcal{F}_t$ , namely

$$\mathbb{E}[B_t^3|\mathcal{F}_t] = \mathbb{E}[(B_t - B_s + B_s)^3|\mathcal{F}_t] = \mathbb{E}[(B_t - B_s)^3 + 3(B_t - B_s)^2B_s] + 3(B_t - B_s)B_s^2 + B_s^3|\mathcal{F}_t] = \mathbb{E}[(B_t - B_s)^3] + 3B_s\mathbb{E}[(B_t - B_s)^2] + 3B_s^2\mathbb{E}[B_t - B_s] + B_s^3.$$

 $B_t - B_s \sim N(0, \sqrt{t-s})$  and since the normal distribution is symmetric all odd moments is zero, so

$$\mathbb{E}[B_t^3|\mathcal{F}_t] = 0 + 3B_s(t-s)^2 + 0 + B_s^3 = B_s^3 + 3(t-s)B_s.$$

### 12.3

Show that the following processes are martingales with respect to  $\mathcal{F}_t$ , the filtration generated by theone dimensional Brownian motion  $B_t$ 

- 1.  $B_t^3 3tB_t$
- 2.  $B_t^4 6tB_t^2 + 3t^2$ .

*Proof.* 1) From Exercise 12.2 we have that

$$\mathbb{E}[B_t^3|\mathcal{F}_t] = B_s^3 + 3(t-s)B_s.$$

Using this together with the fact that  $B_t$  is an  $\mathcal{F}_t$ -martingale we get

$$\mathbb{E}[B_t^3 - 3tB_t | \mathcal{F}_t] = B_s^3 + 3(t-s)B_s - 3tB_s = B_s^3 - 3sB_s$$

which proves the martingale property of  $B_t^3 - 3tB_t$  with respect to  $\mathcal{F}_t$ .

2) We start by computing  $\mathbb{E}[B_t^4|\mathcal{F}_s]$ , and as in Exercise 12.2 we do this by separating  $B_t$  in a part that is measurable with respect to  $\mathcal{F}_s$  and part that is independent of  $\mathcal{F}_s$ 

$$\mathbb{E}[B_t^4|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s + B_s)^4|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^4 + 4(B_t - B_s)^3B_s + 6(B_t - B_s)^2B_s^2 + 4(B_t - B_s)B_s^3 + B_s^4|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^4] + 4B_s\mathbb{E}[(B_t - B_s)^3] + 6B_s^2\mathbb{E}[(B_t - B_s)^2] + 4B_s^3\mathbb{E}[B_t - B_s] + B_s^4.$$

Recall that  $B_t - B_s \sim N(0, \sqrt{t-s})$  so we may write  $B_t - B_s = \sqrt{T-s}X$  where  $X \sim N(0, 1)$  so we may rewrite our expression as

$$\mathbb{E}[B_t^4|\mathcal{F}_s] = (t-s)^2 \mathbb{E}[X^4] + 4B_s \sqrt{t-s}^3 \mathbb{E}[X^3] + 6B_s^2(t-s)\mathbb{E}[X^2] + 4B_s^3 \sqrt{t-s}\mathbb{E}[X] + B_s^4$$

and since all odd moments of the standard normal distribution is zero and the second moment is one we have

$$\mathbb{E}[B_t^4|\mathcal{F}_s] = (t-s)^2 \mathbb{E}[X^4] + 6B_s^2(t-s) + B_s^4.$$

To evaluate  $\mathbb{E}[X^4]$  we use the *moment generating function* of the standard normal distribution

$$\Psi_X(u) = \mathbb{E}[e^{uX}] = e^{u^2/2}$$

and use the result that the *n*'th derivative of  $\Psi_X(u)$  evaluated in u = 0 is the *n*'th moment of X. The fourth derivative of  $\Psi_X(u)$  is

$$\Psi_X^{(n)}(u) = (3 + 6u^2 + u^4)\Psi_X(u)$$

and since  $\Psi_X(0) = 1$  we get  $\Psi_X^{(4)}(0) = \mathbb{E}[X^4] = 3$ . From this we get

$$\mathbb{E}[B_t^4|\mathcal{F}_s] = (t-s)^2 \mathbb{E}[X^4] + 6B_s^2(t-s) + B_s^4 = 3(t-s)^2 + 6B_s^2(t-s) + B_s^4.$$

We now may derive the martingale property of  $B_t^4 - 6tB_t^2 + 3t^2$ , using the fact that  $B_t^2 - t$  in an  $\mathcal{F}_t$ -martingale,

$$\begin{split} \mathbb{E}[B_t^4 - 6tB_t^2 + 3t^2|\mathcal{F}_s] &= 3(t-s)^2 + 6B_s^2(t-s) + B_s^4 - 6t\mathbb{E}[B_t^2|\mathcal{F}_s] + 3t^2 \\ &= 3(t-s)^2 + 6B_s^2(t-s) + B_s^4 - 6t\mathbb{E}[B_t^2 - t + t|\mathcal{F}_s] + 3t^2 \\ &= 3(t-s)^2 + 6B_s^2(t-s) + B_s^4 - 6t(B_s^2 - s + t) + 3t^2 \\ &= 3t^2 - 6ts + 3s^2 + 6B_s^2t - 6sB_s^2 + B_s^4 - 6tB_s^2 + 6ts - 6t^2 + 3t^2 \\ &= B_s^4 - 6sB_s^2 + 3s^2, \end{split}$$

hence  $B_t^4 - 6tB_t^2 + 3t^2$  is an  $\mathcal{F}_t$ -martingale

# Chapter 13

### 13.1

Let  $\{t_i\}_{i=0}^\infty$  be an increasing sequence of scalars and define  $t_i^*$  such that  $t_i < t_i^* \le t_{i+1}.$  Furthermore, let

$$\tilde{S}_n = \sum_{i=0}^{n-1} B_{t_i^*} (B_{t_{i+1}} - B_{t_i}).$$

where  $B_{t_i}$  is the discrete Brownian motion.

Check that  $\tilde{S}_k$ ,  $0 \le k \le n$  is not a martingale with respect to the filtration generated by B.

*Proof.* We only chek the martingale property of  $\tilde{S}_k$ .

$$\begin{split} \mathbb{E}[\tilde{S}_{k}|\mathcal{F}_{k-1}] &= \mathbb{E}[\sum_{i=0}^{k-1} B_{t_{i}^{*}}(B_{t_{i+1}} - B_{t_{i}})|\mathcal{F}_{k-1}] = \{B_{t_{i}^{*}}(B_{t_{i+1}} - B_{t_{i}}) \\ & \text{are } \mathcal{F}_{k-1}\text{-measurable for } i \leq k-2\} = \underbrace{\sum_{i=0}^{k-2} B_{t_{i}^{*}}(B_{t_{i+1}} - B_{t_{i}})}_{=\tilde{S}_{k-1}} \\ & + \mathbb{E}[B_{t_{k-1}^{*}}(B_{t_{k}} - B_{t_{k-1}})|\mathcal{F}_{k-1}] = \{\mathbb{E}[B_{t_{k-1}}(B_{t_{k}} - B_{t_{k-1}})|\mathcal{F}_{k-1}] = 0\} \\ &= \tilde{S}_{k-1} + \mathbb{E}[B_{t_{k-1}^{*}}(B_{t_{k}} - B_{t_{k-1}})|\mathcal{F}_{k-1}] - \mathbb{E}[B_{t_{k-1}}(B_{t_{k}} - B_{t_{k-1}})|\mathcal{F}_{k-1}] \\ &= \tilde{S}_{k-1} + \mathbb{E}[(B_{t_{k-1}^{*}} - B_{t_{k-1}})(B_{t_{k}} - B_{t_{k-1}})|\mathcal{F}_{k-1}] = \tilde{S}_{k-1} + (t_{k-1}^{*} - t_{k-1}) \\ &\neq \tilde{S}_{k-1} \end{split}$$

for any  $t_{k-1} < t_{k-1}^* \le t_k$  hence  $\tilde{S}_k$  is not a martingale with respect to the filtration generated by the Brownian motion B.

13.2

Let  $B_t$  be a Brownian motion and let  $X_t$  be the stochastic integral

$$X_t = \int_0^t e^{s-t} dB_s$$

- 1. Determine the expectation  $\mathbb{E}[X_t]$  and the variance  $V(X_t)$  of  $X_t$ .
- 2. Show that the random variable

$$W_t = \sqrt{2(t+1)X_{\log(t+1)/2}}$$

has distribution  $W_t \sim N(0, t)$ .

*Proof.* 1) By part (vi) of Proposition 13.11, that defines properties of then Ito integral, we have that since the integrand,  $e^{s-t}$ , of the stochastic integral is deterministic, the stochastic integral is normally distributed as

$$X_t \sim N\left(0, \int_0^t (e^{s-t})^2 ds\right) = N\left(0, \int_0^t e^{2(s-t)} ds\right) = N\left(0, \frac{1}{2}(1-e^{-2t})\right).$$

Hence  $X_t$  has the distribution  $X_t \sim N\left(0, \frac{1}{2}(1-e^{-2t})\right)$ .

2) From the first part of the exercise, we know that  $X_t \sim N\left(0, \frac{1}{2}(1-e^{-2t})\right)$ . For a normally distributed random variable  $Y \sim N(0, \sigma)$  it holds that  $cY \sim N(0, c^2\sigma^2)$  hence

$$W_t \sim N\left(0, \left(\sqrt{2(t+1)}\right)^2 \frac{1}{2} (1 - e^{-2\log(t+1)/2})\right) = N\left(0, (t+1)(1 - e^{-\log(t+1)})\right)$$
$$= N\left(0, (t+1)(1 - \frac{1}{t+1})\right) = N\left(0, (t+1)\frac{t}{t+1}\right) = N\left(0, t\right).$$

And the proof is done.

## Chapter 15

## 15.3

Let B be a Brownian motion. Find  $z \in \mathbb{R}$  and  $\varphi(s, \omega) \in \mathcal{V}$  such that

$$F(\omega) = z + \int_0^T \varphi(s, \omega) dB_s$$

in the following cases

1.  $F(\omega) = B_T^3(\omega).$ 2.  $F(\omega) = \int_0^T B_s^3 ds.$ 3.  $F(\omega) = e^{T/2} \cosh(B_T(\omega)) = e^{T/2} \frac{1}{2} (e^{B_T(\omega)} + e^{-B_T(\omega)})$  Proof. 1) We get

$$d(B_t^3) = 3B_t^2 dB_t + \frac{1}{2}6B_t dt$$

and since

$$d(tB_t) = tdB_t + B_t dt$$

we have  $tB_t = \int_0^t s dB_s + \int_0^t B_s ds$  which may be rewritten using  $tB_t = \int_0^t t dB_s$ to get  $\int_0^t B_s ds = \int_0^t (t-s) dB_s$ . We may now write the  $B_T^3$  as

$$B_T^3 = z + \int_0^T 3(B_t^2 + (T - t))dB_t$$

where  $z = \mathbb{E}[B_T^3] = 0$ . Hence  $\varphi(s, \omega) = 3(B_s(\omega)^2 + (T - s))$ 2)

$$d(TB_T^3) = T(3B_T^2 dB_T + 3B_T dT) + B_T^3 dT$$

hence

$$TB_T^3 = -z + \int_0^T s 3B_s^2 dB_s + \int_0^T 3sB_s ds + \int_0^T B_s^3 ds,$$

for some  $z \in \mathbb{R}$ . Rewriting the expression gives

$$\int_0^T B_s^3 ds = z + T B_T^3 - \int_0^T s 3 B_s^2 dB_s - \int_0^T 3 s B_s ds$$

which by problem 1) gives

$$\int_0^T B_s^3 ds = z + \int_0^T \left( 3T(B_s^2 + (T-s)) - s3B_s^2 \right) dB_s - \int_0^T 3sB_s ds.$$

We need to rewrite  $\int_0^T 3sB_s ds$  on a form that is with respect to  $dB_s$  instead of ds. Study

$$d(T^2B_T) = 2TB_T dT + T^2 dB_T$$

hence  $T^2 B_T = \int_0^T 2sB_s ds + \int_0^T s^2 dB_s$  and by writing  $T^2 B_T = \int_0^T T^2 dB_s$  we get

$$\int_0^T sB_s ds = \frac{1}{2} \int_0^T (T^2 - s^2) dB_s,$$

hence

$$\int_0^T B_s^3 ds = z + \int_0^T \left( 3T(B_s^2 + (T-s)) - s3B_s^2 \right) dB_s - \frac{3}{2} \int_0^T (T^2 - s^2) dB_s.$$

We may now write the  $\int_0^T B_s^3 ds$  as

$$\int_0^T B_s^3 ds = z + \int_0^T \left( 3T(B_s^2 + (T-s)) - s3B_s^2 - \frac{3}{2}(T^2 - s^2) \right) dB_s$$

where  $z = \mathbb{E}[\int_{0}^{T} B_{s}^{3} ds] = \int_{0}^{T} \mathbb{E}[B_{s}^{3}] ds = 0$ . Hence  $\varphi(s, \omega) = 3T(B_{s}(\omega)^{2} + (T - s)) - s3B_{s}(\omega)^{2} - \frac{3}{2}(T^{2} - s^{2})$ . 3) We notice that since

$$\begin{split} &d(e^{B_t-t/2}) = e^{B_t-t/2} dB_t \\ &d(e^{-B_t+t/2}) = -e^{-B_t+t/2} dB_t, \end{split}$$

we may write

$$\begin{split} e^{T/2} \frac{1}{2} (e^{B_T(\omega)} + e^{-B_T(\omega)}) &= e^{T/2} \frac{1}{2} (e^{T/2} e^{B_T(\omega) - T/2} + e^{-T/2} e^{-B_T(\omega) + T/2}) \\ &= z + e^{T/2} \frac{1}{2} (e^{T/2} \int_0^T e^{B_s - s/2} dB_s - e^{-T/2} \int_0^T e^{-B_s + s/2} dB_s) \\ &= z + \frac{1}{2} (e^T \int_0^T e^{B_s - s/2} dB_s - \int_0^T e^{-B_s + s/2} dB_s) \\ &= z + \int_0^T \frac{e^T e^{B_s - s/2} - e^{-B_s + s/2}}{2} dB_s \end{split}$$

where z is

$$z = \mathbb{E}[F] = e^{T/2} \frac{1}{2} \mathbb{E}[e^{B_T}] + \mathbb{E}[e^{-B_T}] = e^{T/2} \frac{1}{2} \left( e^{T/2} \underbrace{\mathbb{E}[e^{B_T - T/2}]}_{=1} + e^{-T/2} \underbrace{\mathbb{E}[e^{-B_T + T/2}]}_{=1} \right) = e^{T/2} \frac{1}{2} \left( e^{T/2} + e^{-T/2} \right) = \frac{1}{2} (e^T + 1).$$

### 15.5

Let  $X_t$  be a generalized geometric grownian motion given by

$$dX_t = \alpha_t X_t dt + \beta_t X_t dB_t \tag{3}$$

where  $\alpha_t$  and  $\beta_t$  are bounded deterministic functions and B is a Brownian motion.

1. Find an explicit expression for  $X_t$  and compute  $\mathbb{E}[X_t]$ .

2. Find 
$$z \in \mathbb{R}$$
 and  $\varphi(t, \omega) \in \mathcal{V}$  such that  $X(T, \omega) = z + \int_0^t \varphi(s, \omega) dB_s(\omega)$ .

*Proof.* 1) Let  $X_t = e^{\int_0^t \alpha_s ds} Y_t$ , the differential of  $X_t$  is

$$dX_t = \alpha_t e^{\int_0^t \alpha_s ds} Y_t dt + e^{\int_0^t \alpha_s ds} dY_t = \alpha_t X_t dt + e^{\int_0^t \alpha_s ds} dY_t$$

for this expression to be equal to (3) we must have

$$e^{\int_0^t \alpha_s ds} dY_t = \beta_t \underbrace{e^{\int_0^t \alpha_s ds} Y_t}_{=X_t} dB_t$$

which holds if  $dY_t = \beta Y_t dB_t$  hence  $Y_t$  is an exponential martingale given by

$$Y_t = y_0 e^{\int_0^t \beta_t dB_t - \frac{1}{2} \int_0^t \beta_t^2 dt}$$

where  $y_0 = 1$  since  $X_0 = 1$ . This gives the following expression for  $X_t$ 

$$X_t = e^{\int_0^t \beta_t dB_t + \int_0^t (\alpha_t - \frac{1}{2}\beta_t)^2 d}$$

Using the martiongale property of  $Y_t$  gives

$$\mathbb{E}[X_t] = \mathbb{E}[e^{\int_0^t \alpha_s ds} Y_t] = e^{\int_0^t \alpha_s ds} \underbrace{\mathbb{E}[Y_t]}_{=Y_0} = e^{\int_0^t \alpha_s ds}.$$

2) In part 1) it was shown that  $X_T = e^{\int_0^T \alpha_s ds} Y_T$  where  $Y_T$  is the exponential martingale with SDE  $dY_t = \beta_t Y_t dB_t$  hence  $X_T$  may be written as

$$X_T = e^{\int_0^T \alpha_s ds} Y_T = e^{\int_0^T \alpha_s ds} (1 + \int_0^T \beta_s Y_s dB_s) = \underbrace{e^{\int_0^T \alpha_s ds}}_{=z} + \int_0^T e^{\int_0^T \alpha_s ds} \beta_s Y_s dB_s$$

and  $\varphi(s,\omega) = e^{\int_0^T \alpha_s ds} \beta_s Y_s(\omega)$ . We need to show that  $\varphi(s,\omega) \in \mathcal{V}$ , the criterias are given in Definition 13.1. Part 1 and 2 of Definiton 13.1 is showed by noticing that  $\varphi_s$  is the product of the processes  $e^{\int_0^T \alpha_u du} \beta_s$  which is deterministic and therefore universally measurable, and  $Y_s$  which is the exponential martingale and therefore fulfill the conditions 1) and 2). The product of these two processes does also fulfill criterias 1) and 2). Condition 3) of Definition 13.1 is shown by using that  $\beta_t$  is bounded so that there is a constant  $0 < K < \infty$  such that  $|\beta_t| < K$  for every  $t \in [0, T]$ ,

$$\mathbb{E}\Big[\int_0^T (\beta_t e^{\int_0^T \alpha_s ds} Y_t)^2 dt\Big] = e^{\int_0^T \alpha_s ds} \mathbb{E}\Big[\int_0^T \beta_t^2 Y_t^2 dt\Big] \le \{|\beta| < K\}$$
(4)

$$\leq \underbrace{e^{\int_0^T \alpha_s ds} K^2}_{\triangleq_{\Gamma}} \mathbb{E}\left[\int_0^T Y_t^2 dt\right] = \{\text{Fubini}\} = \Gamma \int_0^T \mathbb{E}\left[Y_t^2\right] dt.$$
(5)

 $Y_t^2 = e^{\int_0^t 2\beta_s dB_s - \int_0^t \beta_s^2 ds}$  can be written as the product of an exponential martingale and a deterministic function as  $Y_t^2 = e^{\int_0^t 2\beta_s dB_s - \frac{1}{2}\int_0^t (2\beta)_s^2 ds} e^{\int_0^t \beta_s^2 ds}$  so that (4) becomes

$$\begin{split} \mathbb{E}\Big[\int_{0}^{T} (\beta_{t}e^{\int_{0}^{T}\alpha_{s}ds}Y_{t})^{2}dt\Big] &\leq \Gamma \int_{0}^{T} \mathbb{E}\Big[e^{\int_{0}^{t}2\beta_{s}dB_{s}-\frac{1}{2}\int_{0}^{t}(2\beta_{s})^{2}ds}e^{\int_{0}^{t}\beta_{s}^{2}ds}\Big]dt \\ &= \Gamma \int_{0}^{T}e^{\int_{0}^{t}\beta_{s}^{2}ds}\mathbb{E}\Big[e^{\int_{0}^{t}2\beta_{s}dB_{s}-\frac{1}{2}\int_{0}^{t}(2\beta_{s})^{2}ds}\Big]dt = \Gamma \int_{0}^{T}e^{\int_{0}^{t}\beta_{s}^{2}ds}dt \\ &\leq \Gamma \int_{0}^{T}e^{\int_{0}^{t}K^{2}ds}dt \leq \Gamma \int_{0}^{T}e^{K^{2}t}dt = \Gamma \int_{0}^{T}e^{K^{2}t}dt = \Gamma \frac{e^{K^{2}T}-1}{K^{2}} < \infty. \end{split}$$

Hence  $\varphi_s \in \mathcal{V}$ 

## 15.6

Let  $f(t) = e^{t^2/2} - 1$  and let B be a brownian motion on the probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \ge t\}, \mathbb{P})$ . Show that there exists another Brownian motion  $\tilde{B}$  such that

$$X_t = \int_0^{f(t)} \frac{1}{\sqrt{1+s}} dB_s = \int_0^t \sqrt{s} d\tilde{B}_s$$

*Proof.* The quadratic variation of  $X_t$  is

$$\begin{split} \langle X \rangle_t &= \int_0^{f(t)} \left( \frac{1}{\sqrt{1+s}} \right) ds = \int_0^{e^{t^2/2} - 1} \frac{1}{1+s} ds = \log \left( 1 + (e^{t^2/2 - 1}) \right) \\ &= t^2/2 = \int_0^t s ds. \end{split}$$

By Theorem 15.4 there exists an extension  $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t, t \geq t\}, \hat{\mathbb{P}})$  of  $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq t\}, \mathbb{P})$  on which there is a Brownian motion  $\tilde{B}$  such that

$$X_t = \int_0^t s d\tilde{B}_s$$

which solves the problem at hand.

## Chapter 16

## 16.1

Let  $X_t$  solve the SDE

$$dX_t = (\alpha X_t + \beta)dt + (\sigma X_t + \gamma)dB_t X_0 = 0$$

where  $\alpha$ ,  $\beta$ ,  $\sigma$  and  $\gamma$  are constants and B is a Brownian motion. Furthermore, let  $S_t = e^{(\alpha - \sigma^2/2)t + \sigma B_t}$ .

1. derive the SDE satisfied by  $S_t^{-1}$ .

2. Show that  $d(X_t S_t^{-1}) = (\beta - \sigma \gamma) S_t^{-1} dt + \gamma S_t^{-1} dB_t$ .

3. Derive the explicit form of  $X_t$ .

*Proof.* 1) Let  $f(t,x) = e^{-(\alpha - \sigma^2/2)t - \sigma x}$  then the SDE of  $S_t^{-1}$  is

$$\begin{split} dS_{t}^{-1} &= \frac{\partial}{\partial t} f(t, B_{t}) dt + \frac{\partial}{\partial x} f(t, B_{t}) dB_{t} + \frac{1}{2} \frac{\partial^{2}}{\partial x^{2}} f(t, B_{t}) dt \\ &= -(\alpha - \sigma^{2}/2) e^{-(\alpha - \sigma^{2}/2)t - \sigma x} dt - \sigma e^{-(\alpha - \sigma^{2}/2)t - \sigma x} dB_{t} + \frac{1}{2} \sigma^{2} e^{-(\alpha - \sigma^{2}/2)t - \sigma x} dt \\ &= -(\alpha - \sigma^{2}) S_{t}^{-1} dt - \sigma S_{t}^{-1} dB_{t}. \\ 2) \\ d(X_{t}S_{t}^{-1}) &= X_{t} dS_{t}^{-1} + s_{t}^{-1} dX_{t} + \langle X, S^{-1} \rangle = X_{t} (-(\alpha - \sigma^{2}) S_{t}^{-1} dt - \sigma S_{t}^{-1} dB_{t}) \\ &+ S_{t}^{-1} ((\alpha X_{t} + \beta) dt + (\sigma X_{t} + \gamma) dB_{t}) + (-\sigma S_{t}^{-1}) (\sigma X_{t} + \gamma) dt \\ &= -(\alpha - \sigma^{2}) X_{t} S_{t}^{-1} dt - \sigma X_{t} S_{t}^{-1} dB_{t} + \alpha X_{t} S_{t}^{-1} dt \\ &+ \beta S_{t}^{-1} dt + \sigma X_{t} S_{t}^{-1} dB_{t} + \gamma S_{t}^{-1} dB_{t} - \sigma^{2} X_{t} S_{t}^{-1} dt - \sigma \gamma S_{t}^{-1} dt \\ &= (-(\alpha - \sigma^{2}) + \alpha - \sigma^{2}) X_{t} S_{t}^{-1} dt + (-\sigma + \sigma) X_{t} S_{t}^{-1} dB_{t} + (\beta - \gamma \sigma) S_{t}^{-1} dt \\ &+ \gamma S_{t}^{-1} dB_{t} = (-(\alpha - \sigma^{2}) + \alpha - \sigma^{2}) X_{t} S_{t}^{-1} dt + (-\sigma + \sigma) X_{t} S_{t}^{-1} dB_{t} \\ &+ (\beta - \gamma \sigma) S_{t}^{-1} dt + \gamma S_{t}^{-1} dB_{t} = (\beta - \gamma \sigma) S_{t}^{-1} dt + \gamma S_{t}^{-1} dB_{t}. \end{split}$$

3) From 2) we get that

$$X_t S_t^{-1} = \left(\beta - \gamma\sigma\right) \int_0^t S_s^{-1} ds + \gamma \int_0^t S_s^{-1} dB$$

multiplying both sides by  $\mathcal{S}_t$  and express  $\mathcal{S}_t$  on its explicit form gives

$$X_t = e^{(\alpha - \sigma^2/2)t + \sigma B_t} \Big( (\beta - \gamma \sigma) \int_0^t e^{-(\alpha - \sigma^2/2)s - \sigma B_s} ds + \gamma \int_0^t e^{-(\alpha - \sigma^2/2)s - \sigma B_s} dB_s \Big).$$

# **Additional Exercises**

The Additional Exercises problem formulations may be found on the course webpage.

## $\mathbf{A1}$

Compute the stochastic differential dz when

- 1.  $Z_t = e^{\alpha t}, \ \alpha \in \mathbb{R}, \ t \in [0, \infty).$
- 2.  $Z_t = \int_0^t g(s) dB_s$  where g(s) is an adapted stochastic process.
- 3.  $Z_t = e^{\alpha B_t}$ .
- 4.  $Z_t = e^{\alpha X_t}$  where  $dX_t = \mu dt + \sigma dB_t$ .
- 5.  $Z_t = X_t^2$  where  $dX_t = \mu X_t dt + \sigma X_t dB_t$ .

*Proof.* 1) Since  $\alpha t$  is of first variation,  $dZ_t = \alpha e^{\alpha t} dt = \alpha Z_t dt$ .

2)  $dZ_t = g(t)dB_t$  since it is the differential of an integral.

3) Let  $f(x) = e^{\alpha x}$  then, using the Ito formula, since  $\alpha B_t$  has quadratic variation, we get

$$dZ_t = f'(B_t)dB_t + \frac{1}{2}f''(B_t)d\langle B \rangle_t = \alpha e^{\alpha B_t}dB_t + \frac{\alpha^2}{2}e^{\alpha B_t}dt$$
$$= \alpha Z_t dB_t + \frac{\alpha^2}{2}Z_t dt.$$

4) Using the same notation as in 3) we get

$$dZ_t = f'(\alpha X_t) dX_t + \frac{1}{2} f''(\alpha X_t) d\langle X \rangle_t = \alpha e^{\alpha X_t} (\mu dt + \sigma dB_t) + \frac{\alpha^2}{2} e^{\alpha X_t} \sigma^2 dt$$
$$= \alpha \sigma Z_t dB_t + (\alpha \mu + \frac{(\alpha \sigma)^2}{2}) Z_t dt.$$
  
5) Let  $f(x) = x^2$ , we get

$$dZ_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X \rangle_t = 2X_t dX_t + \frac{1}{2}2d\langle X \rangle_t$$
  
=  $2X_t(\mu X_t dt + \sigma X_t dB_t) + \sigma^2 X_t^2 dt = 2\mu X_t^2 dt + 2\sigma X_t^2 dB_t + \sigma^2 X_t^2 dt$   
=  $(2\mu + \sigma^2)Z_t dt + 2\sigma Z_t dB_t.$ 

## $\mathbf{A2}$

Compute te stochastic differential for Z when  $Z_t = 1/X_t$  where  $X_t$  has the differential

$$dX_t = \alpha X_t dt + \sigma X_t dB_t.$$

*Proof.* Let f(x) = 1/x and hence  $f'(x) = -1/x^2$  and  $f''(x) = 2/x^3$ . By the Ito formula we get

$$dZ_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X \rangle_t = -\frac{1}{X_t^2}dX_t + \frac{1}{2}\frac{2}{X_t^2}d\langle X \rangle_t$$
$$= -\frac{\alpha X_t dt + \sigma X_t dB_t}{X_t^2} + \frac{1}{X_t^3}\sigma^2 X_t^2 dt = -\frac{\alpha dt + \sigma dB_t}{X_t} + \frac{\sigma^2}{X_t}dt$$
$$= (-\alpha + \sigma^2)Z_t dt - \sigma Z_t dB_t.$$

## $\mathbf{A5}$

Let B be a Brownian motion and  $\{\mathcal{F}_t\}$  be the filtration generated by B. Show by using stochastic calculus that the following processes are martingales.

- 1.  $B_t^2 t$ .
- 2.  $e^{\lambda B_t \frac{\lambda^2 t}{2}}$ .

*Proof.* If the Ito differential only has an  $dB_t$ -part, i.e. that differential looks on the form  $0dt + (...)dB_t$ , and the integrand is a member of the class  $\mathcal{V}$  defined in Definition 13.1, then we may be certain that the process is a martingale by Proposition 13.11. 1)

$$d(B_t^2 - t) = 2B_t dB_t + \frac{1}{2}2dt - dt = 2B_t dB_t$$

and since

$$\mathbb{E}[\int_0^t B_s^2 ds] = \{\text{Fubini}\} = \int_0^t \mathbb{E}[B_s^2] ds = \int_0^t s ds = \frac{t^2}{2} < \infty,$$

 $2B_t \in \mathcal{V}$ , and hence  $B_t^2 - t$  is a martingale. 2)

$$\begin{split} d(e^{\lambda B_t - \frac{\lambda^2 t}{2}}) &= -\frac{\lambda^2}{2} e^{\lambda B_t - \frac{\lambda^2 t}{2}} dt + \lambda e^{\lambda B_t - \frac{\lambda^2 t}{2}} dB_t + \frac{1}{2} \lambda^2 e^{\lambda B_t - \frac{\lambda^2 t}{2}} dt \\ &= \lambda e^{\lambda B_t - \frac{\lambda^2 t}{2}} dB_t. \end{split}$$

And since

$$\mathbb{E}\left[\int_{0}^{t} \left(e^{\lambda B_{s}-\frac{\lambda^{2}s}{2}}\right)^{2} ds\right] = \mathbb{E}\left[\int_{0}^{t} e^{2\lambda B_{s}-\lambda^{2}s} ds\right] = \{\text{Fubini}\} = \int_{0}^{t} \mathbb{E}\left[e^{2\lambda B_{s}-\lambda^{2}s}\right] ds$$
$$= \int_{0}^{t} \mathbb{E}\left[e^{2\lambda B_{s}}\right] e^{-\lambda^{2}s} ds = \{B_{s} \sim N(0,s)\} = \int_{0}^{t} e^{4\lambda^{2}s} e^{-\lambda^{2}s} ds < \infty, \ 0 \le t < \infty$$

hence  $e^{\lambda B_t - \frac{\lambda^2 t}{2}}$  is a matingale.

### A6

Check whether the following processes are martingales with respect to the filtration generated by  $B_t$ .

- 1.  $X_t = B_t + 4t$
- 2.  $X_t = B_t^2$
- 3.  $X_t = t^2 B_t 2 \int_0^t r B_r dr$
- 4.  $X_t = B_t^{(1)} B_t^{(2)}$  where  $(B_t^{(1)}, B_t^{(2)})$  is a two dimensional Brownian motion.

*Proof.* The proofs may (at least) be done in two ways, one by checking the martingale property of the process using a time s < t, and the other is to do the stochastic differential of the process and the use the Martingale Representation Theorem (Theorem 15.3) to conclude whether the process is a martingale or not. We will do both. When using the Martingale Representation Theorem we should also prove that the integrand is a member of the class  $\mathcal{V}$ , which we omit in the solutions.

1)  $\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[B_t + 4t|\mathcal{F}_s] = B_s + 4t \neq B_s + 4s$  hence  $B_t + 4t$  is not an  $\mathcal{F}_t$ -martingale.  $dX_t = dB_t + 4dt \neq g(s, \omega)dB_t \text{ hence } B_t + 4t \text{ is not an } \mathcal{F}_t\text{-martingale.}$   $2) \mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[B_t^2|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s + B_s)^2|\mathcal{F}_s] = \mathbb{E}[(B_t - B_s)^2|\mathcal{F}_s] + \mathbb{E}[B_s^2|\mathcal{F}_s] = t - s + B_s^2 \neq B_s^2 \text{ hence } B_t^2 \text{ is not an } \mathcal{F}_t\text{-martingale.}$  $dX_t = 2B_t dB_t + dt \neq g(s, \omega) dB_t$  hence  $B_t^2$  is not an  $\mathcal{F}_t$ -martingale. 3)

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[t^2 B_t - 2\int_0^t r B_r dr |\mathcal{F}_s] = t^2 B_s - 2\mathbb{E}[\int_0^t r B_r dr |\mathcal{F}_s]$$
  
=  $t^2 B_s - 2\int_0^t r\mathbb{E}[B_r|\mathcal{F}_s]dr = t^2 B_s - 2\int_0^s r\underbrace{\mathbb{E}[B_r|\mathcal{F}_s]}_{=B_r}dr - 2\int_s^t r\underbrace{\mathbb{E}[B_r|\mathcal{F}_s]}_{=B_s}dr$   
=  $t^2 B_s - 2\int_0^s r B_r dr - 2B_s\int_s^t r dr = s^2 B_s - 2\int_0^s r B_r dr = X_s$ 

hence  $t^2 B_t - 2 \int_0^t r B_r dr$  is an  $\mathcal{F}_t$ -martingale.  $dX_t = 2t B_t dt + t^2 dB_t - 2t B_t dt = t^2 dB_t$  hence  $t^2 B_t - 2 \int_0^t r B_r dr$  is an  $\mathcal{F}_t$ martingale.

 $\begin{array}{l} \text{Matteringate.} \\ 4) \ \mathbb{E}[X_t | \mathcal{F}_s] = \mathbb{E}[B_t^{(1)} B_t^{(2)} | \mathcal{F}_s] = \{\text{independent}\} = \mathbb{E}[B_t^{(1)} | \mathcal{F}_s] \mathbb{E}[B_t^{(2)} | \mathcal{F}_s] = \\ B_s^{(1)} B_s^{(2)} \text{ hence } B_t^{(1)} B_t^{(2)} \text{ is an } \mathcal{F}_t \text{-martingale.} \\ dX_t = B_t^{(1)} dB_t^{(2)} + B_t^{(2)} dB_t^{(1)} + d\langle B^{(1)}, B_t^{(2)} \rangle = B_t^{(1)} dB_t^{(2)} + B_t^{(2)} dB_t^{(1)} \text{ hence } \\ B_t^{(1)} B_t^{(2)} \text{ is an } \mathcal{F}_t \text{-martingale.} \end{array}$ 

### A7

Let X be the solution to the SDE

$$dX_t = \alpha X_t dt + \sigma dB_t, \ X_0 = x_0$$

where  $\alpha, \sigma, x_0$  are constants and B is a brownian motion.

- 1. determine  $\mathbb{E}[X_t]$ .
- 2. Determine  $V(X_t)$ .

3. Determine the solution to true SDE.

Proof. 1)

$$\mathbb{E}[X_t] = \mathbb{E}[x_0 + \alpha \int_0^t X_s ds + \sigma \int_0^t dB_s] = x_0 + \alpha \mathbb{E}[\int_0^t X_s ds] + \sigma \underbrace{\mathbb{E}[\int_0^t dB_s]}_{=0}$$
$$= x_0 + \alpha \mathbb{E}[\int_0^t X_s ds]$$

which leads to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[X_t] = \alpha \mathbb{E}[X_t]$$
$$\mathbb{E}[X_0] = x_0$$

which has the solution  $\mathbb{E}[X_t] = x_0 e^{\alpha t}$ 2)  $V(X_t) = \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2 = \mathbb{E}[X_t^2] - x_0 e^{\alpha t}$  so we need to find an explicit expression for  $\mathbb{E}[X_t^2]$ . Looking at the stochastic differential of  $X_t^2$  we get

$$dX_t^2 = 2X_t dX_t + d\langle X \rangle_t = 2X_t (\alpha X_t dt + \sigma dB_t) + \sigma^2 dt$$
$$= (2\alpha X_t^2 + \sigma^2) dt + 2\sigma X_t dB_t$$

from which we have

$$X_t^2 = x_0^2 + \int_0^t (2\alpha X_t^2 + \sigma^2) dt + 2\sigma \int_0^t X_t dB_t$$

 $\mathbb{E}[X_t^2]$  becomes

$$\mathbb{E}[X_t^2] = \mathbb{E}[x_0^2 + \int_0^t (2\alpha X_t^2 + \sigma^2)dt + 2\sigma \int_0^t X_t dB_t] = x_0^2 + \mathbb{E}[\int_0^t 2\alpha X_t^2 dt] + \sigma^2 t$$
(6)

$$+2\sigma \underbrace{\mathbb{E}[\int_{0}^{t} X_{t} dB_{t}]}_{=0 \text{ (Mg)}} = \{\text{Fubini}\} = x_{0}^{2} + \int_{0}^{t} 2\alpha \mathbb{E}[X_{t}^{2}] dt + \sigma^{2} t.$$

Let  $g(t) = \mathbb{E}[X_t^2]$ , the problem (6) leads to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}g(t) = 2\alpha g(t) + \sigma^2.$$

Consider the function  $e^{-2\alpha t}g(t)$ . Differentiating  $e^{-2\alpha t}g(t)$  we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big( e^{-2\alpha t} g(t) \Big) = -2\alpha g(t) e^{-2\alpha t} + e^{-2\alpha t} \frac{\mathrm{d}}{\mathrm{d}t} g(t) = e^{-2\alpha t} \sigma^2.$$

Integrating both sides and multiplying both sides by  $e^{2\alpha t}$  we get

$$g(t) = e^{2\alpha t} \left( c + \sigma^2 \int_0^t e^{-2\alpha s} ds \right).$$

c is determined by the initial condition  $g(0) = x_0^2$  as  $c = x_0^2$ . And the formula for  $\mathbb{E}[X_t]$  is

$$\mathbb{E}[X_t] = e^{2\alpha t} \left( x_0^2 + \sigma^2 \int_0^t e^{-2\alpha s} ds \right) = x_0^2 e^{2\alpha t} + \sigma^2 \frac{e^{2\alpha t} - 1}{2\alpha}$$

so the variance is  $V(X_t) = \mathbb{E}[X_t^2] - (\mathbb{E}[X_t])^2 = \sigma^2 \frac{e^{2\alpha t} - 1}{2\alpha}$ . 3) Taking the differential of  $e^{-\alpha t}X_t$  yields

$$d\left(e^{-\alpha t}X_{t}\right) = -\alpha e^{-\alpha t}X_{t}dt + e^{-\alpha t}dX_{t} = e^{-\alpha t}\sigma dB_{t}.$$
(7)

Integrating both sides of (7) and multiplying both sides with  $e^{\alpha t}$  gives

$$X_t = e^{\alpha t} \left( c + \int_0^t e^{-\alpha t} \sigma dB_t \right)$$

where c is determined by the initial condition  $X_0 = x_0$  to be  $c = x_0$ . The solution of the SDE is therefore given by

$$X_t = e^{\alpha t} \left( x_0 + \int_0^t e^{-\alpha t} \sigma dB_t \right)$$

 $X_t$  is the so called Ornstein Uhlenbeck process.

## A9

Let h(t) be a deterministic function and define the process  $X_t$  as

$$X_t = \int_0^t h(s) dB_s.$$

Show that  $X_t \sim N(0, \int_0^t h^2(s) ds)$  by showing that

$$\mathbb{E}[e^{iuX_t}] = e^{-\frac{u^2}{2}\int_0^t h^2(s)ds}.$$
(8)

*Proof.* Recall that (8) is the characteristic function of  $N(0, \int_0^t h^2(s)ds)$  which is a unique transformation, therefore proving (8) is equal to proving that  $X_t \sim \int_0^t h^2(s)ds$  $N(0,\int_0^t h^2(s)ds).$  Let  $Y_t=e^{iuX_t}.$  Since  $dX_t=h(t)dB_t$  we get

$$dY_t = iuY_t dX_t - \frac{u^2}{2}Y_t d\langle X \rangle_t = iuY_t h(t)dB_t - \frac{u^2}{2}Y_t h^2(t)dt.$$

so  $Y_t = 1 + iu \int_0^t Y_t h(s) dB_s - \frac{u^2}{2} \int_0^t Y_s h^2(s) ds$  since  $Y_0 = 1$ . The expected value of  $Y_t$  is

$$\mathbb{E}[Y_t] = \mathbb{E}[1 + iu \int_0^t Y_t h(s) dB_s - \frac{u^2}{2} \int_0^t Y_s h^2(s) ds] = 1 + iu \underbrace{\mathbb{E}[\int_0^t Y_t h(s) dB_s]}_{=0}$$
(9)

$$-\frac{u^2}{2}\mathbb{E}[\int_0^t Y_s h^2(s)ds] = \{\text{Fubini}\} = 1 - \frac{u^2}{2}\int_0^t \mathbb{E}[Y_s]h^2(s)ds.$$
(10)

Let  $g(t) = \mathbb{E}[Y_t]$  and (9) may be written as the ordinary differential equation

$$\frac{t}{dt}g(t) = -\frac{u^2}{2}h^2(s)g(s)$$
$$g(0) = 1$$

which has the solution  $g(t) = e^{-\frac{u^2}{2} \int_0^t h^2(s) ds}$  hence

$$\mathbb{E}[e^{iuX_t}] = e^{-\frac{u^2}{2}\int_0^t h^2(s)ds}$$

from which we conclude that  $X_t \sim N(0, \int_0^t h^2(s) ds)$ .

A10

Let X, Y satisfy the following system of SDE's

$$dX_t = \alpha X_t dt + Y_t dB_t, \quad X_0 = x_0$$
  
$$dY_t = \alpha Y_t dt - X_t dB_t, \quad Y_0 = y_0$$

- 1. Show that  $R_t = X_t^2 + Y_t^2$  is deterministic.
- 2. Compute  $\mathbb{E}[X_t]$ ,  $\mathbb{E}[Y_t]$  and  $\operatorname{Cov}(X_t, Y_t)$ .

*Proof.* We first calculate the stochastic differentials of  $X_t^2$  and  $Y_t^2$ .

$$dX_t^2 = 2X_t dX_t + d\langle X \rangle_t = 2\alpha X_t^2 dt + 2X_t Y_t dB_t + Y_t^2 dt$$
$$dY_t^2 = 2\alpha Y_t^2 dt - 2Y_t X_t dB_t + X_t^2 dt$$

so we may write  $R_t$  as

$$R_{t} = X_{t}^{2} + Y_{t}^{2} = x_{0} + 2\alpha \int_{0}^{t} X_{t}^{2} ds + 2 \int_{0}^{t} X_{s} Y_{s} dB_{s} + \int_{0}^{t} Y_{s}^{2} ds + y_{0} + 2\alpha \int_{0}^{t} Y_{s}^{2} ds - 2 \int_{0}^{t} Y_{s} X_{s} dB_{s} + \int_{0}^{t} X_{s}^{2} ds = x_{0} + y_{0}$$
(11)  
+ (1 + 2\alpha)  $\int_{0}^{t} X_{t}^{2} ds + (1 + 2\alpha) \int_{0}^{t} Y_{s}^{2} ds = (1 + 2\alpha) \int_{0}^{t} R_{s} ds.$ 

Let  $g(t) = R_t$ , then (11) can be written as the ordinary differential equation

$$\frac{d}{dt}g(t) = (1+2\alpha)g(s)$$
$$g(0) = x_0 + y_0$$

with the solution  $g(t) = (x_0 + y_0)e^{(1+2\alpha)t}$ . It has been shown that  $R_t = (x_0 + y_0)e^{(1+2\alpha)t}$ .

2) rewriting  $X_t$  and  $Y_t$  on integral form we get

$$X_t = x_0 + \alpha \int_0^t X_s ds + \int_0^t Y_s dB_s$$
$$Y_t = y_0 + \alpha \int_0^t Y_s ds - \int_0^t X_s dB_s.$$

we get

$$\mathbb{E}[X_t] = \mathbb{E}[x_0 + \alpha \int_0^t X_s ds + \int_0^t Y_s dB_s] = x_0 + \alpha \mathbb{E}[\int_0^t X_s ds] + \underbrace{\mathbb{E}[\int_0^t Y_s dB_s]}_{=0}$$
$$= x_0 + \alpha \int_0^t \mathbb{E}[X_s] ds$$
$$\mathbb{E}[Y_t] = \mathbb{E}[y_0 + \alpha \int_0^t Y_s ds - \int_0^t X_s dB_s] = y_0 + \alpha \mathbb{E}[\int_0^t Y_s ds] - \underbrace{\mathbb{E}[\int_0^t X_s dB_s]}_{=0}$$
$$= y_0 + \alpha \int_0^t \mathbb{E}[Y_s] ds.$$

This gives two ordinary differential equations solved in the same manner as in part 1) and gives the solutions

$$\mathbb{E}[X_t] = x_0 e^{\alpha t}$$
$$\mathbb{E}[Y_t] = y_0 e^{\alpha t}.$$

Since the covariance is  $\operatorname{Cov}(X_t, Y_t) = \mathbb{E}[X_t Y_t] - \mathbb{E}[X_t]\mathbb{E}[Y_t]$  we need to evalculate  $\mathbb{E}[X_t Y_t]$ . For this purpose, consider the stochastic differential of  $X_t Y_t$ ,

$$dX_tY_t = X_tdY_t + Y_tdX_t + d\langle X, Y \rangle_t = Y_t(\alpha X_tdt + Y_tdB_t) + X_t(\alpha Y_tdt - X_tdB_t) - X_tY_tdt = (2\alpha - 1)X_tY_tdt + Y_t^2dB_t - X_t^2dB_t.$$

Hence 
$$X_t Y_t = x_0 y_0 + (2\alpha - 1) \int_0^t X_s Y_s ds + \int_0^t (Y_s^2 - X_s^2) dB_s$$

and we get

$$\mathbb{E}[X_t Y_t] = \mathbb{E}[x_0 y_0 + (2\alpha - 1) \int_0^t X_s Y_s ds + \int_0^t (Y_s^2 - X_s^2) dB_s]$$
(12)

$$= x_0 y_0 + (2\alpha - 1)\mathbb{E}[\int_0^t X_s Y_s ds] + \underbrace{\mathbb{E}[\int_0^t (Y_s^2 - X_s^2) dB_s]}_{=0} = \{\text{Fubini}\}$$
(13)

$$= x_0 y_0 + (2\alpha - 1) \int_0^t \mathbb{E}[X_s Y_s] ds.$$
 (14)

Let  $g(t) = \mathbb{E}[X_t Y_t]$ , then solving (12) amounts to solving the ordinary differential equation

$$\frac{d}{dt}g(t) = (2\alpha - 1)g(t)$$
$$g(0) = x_0y_0$$

which has the solution  $g(t)=x_0y_0e^{(2\alpha-1)t}$  hence  $\mathbb{E}[X_tY_t]=x_0y_0e^{(2\alpha-1)t}$  and we get

$$\operatorname{Cov}(X_t, Y_t) = \mathbb{E}[X_t Y_t] - \mathbb{E}[X_t] \mathbb{E}[Y_t] = x_0 y_0 e^{(2\alpha - 1)t} - x_0 y_0 e^{2\alpha t} = x_0 y_0 e^{2\alpha t} (e^{-t} - 1).$$

A11

Let X and Y be processes given by the SDE's

$$dX_t = \alpha X_t dB_t^{(1)} + \beta X_t dB_t^{(2)}, \quad X_0 = x_0$$
  
$$dY_t = \gamma Y_t dt + \sigma Y_t dB_t^{(1)}, \quad Y_0 = y_0$$

where  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\sigma$  are constants and  $B^{(1)}$ ,  $B^{(2)}$  are independent Brownian motions. Compute  $\mathbb{E}[X_tY_t]$ .

*Proof.* Start by differentiating  $X_t Y_t$  to get

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + \langle X, Y \rangle_t = X_t (\gamma Y_t dt + \sigma Y_t dB_t^{(1)}) + Y_t (\alpha X_t dB_t^{(1)} + \beta X_t dB_t^{(2)}) + \alpha \sigma X_t Y_t dt = (\gamma + \alpha \sigma) X_t Y_t dt + (\sigma + \alpha) X_t Y_t dB_t^{(1)} + \beta X_t Y_t dB_t^{(2)}.$$

Using the initial condition  $X_0Y_0 = x_0y_0$  we have

$$X_t Y_t = x_0 y_0 + (\gamma + \alpha \sigma) \int_0^t X_s Y_s ds + (\sigma + \alpha) \int_0^s X_s Y_s dB_s^{(1)} + \beta \int_0^t X_s Y_s dB_s^{(2)}$$

so  $\mathbb{E}[X_t Y_t]$  becomes

$$\mathbb{E}[X_t Y_t] = \mathbb{E}[x_0 y_0 + (\gamma + \alpha \sigma) \int_0^t X_s Y_s ds + (\sigma + \alpha) \int_0^s X_s Y_s dB_s^{(1)}$$
(15)

$$+\beta \int_0^t X_s Y_s dB_s^{(2)}] = x_0 y_0 + (\gamma + \alpha \sigma) \mathbb{E}[\int_0^t X_s Y_s ds] + (\sigma + \alpha) \underbrace{\mathbb{E}[\int_0^s X_s Y_s dB_s^{(1)}]}_{=0}$$
(16)

$$+\beta \underbrace{\mathbb{E}\left[\int_{0}^{t} X_{s} Y_{s} dB_{s}^{(2)}\right]}_{=0} = \{ \operatorname{Fubini} \} = x_{0} y_{0} + (\gamma + \alpha \sigma) \int_{0}^{t} \mathbb{E}[X_{s} Y_{s}] ds.$$
(17)

Let  $g(t) = \mathbb{E}[X_t Y_t]$ , then (15) gives the following ordinary differential equation

$$\frac{d}{dt}g(t) = (\gamma + \alpha\sigma)g(t)$$
$$g(0) = x_0y_0$$

which has the solution  $g(t) = x_0 y_0 e^{(\gamma + \alpha \sigma)t}$  hence  $\mathbb{E}[X_t Y_t] = x_0 y_0 e^{(\gamma + \alpha \sigma)t}$ .  $\Box$ 

## References

[1] Boualem Djehiche, (2009). Stochastic Calculus, an Introduction with Applications. Lecture notes.