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The local volatility surface

Introduction

These notes presents a derivation of what is known as *Dupire's formula* by using stochastic calculus. The presentation is formal and intendend to be mathematically relatively non-technical. Our starting point is the generalization of the standard Black-Scholes model where we exchange the constant volatility for a function depending on time and stock price (for the underlying theory and notation used here we refer to Björk [1]). The presentation follows the one given in Jeanblanc et al [2].

Local volatility models

In the standard Black-Scholes model the P-dynamics of the stock price is given by

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t).$$

Under the unique martingale measure Q, where the bank account is used as numeraire, the dynamics of S is given by (we assume throughout these notes that the stock does not pay out any dividends)

$$dS(t) = rS(t)dt + \sigma S(t)dW^Q(t),$$

where W^Q is a Q-Wiener process. In general, by changing measure from P to Q using Girsanov's theorem only changes the drift and not the volatility. If we use the more general model

$$dS(t) = \alpha(t, S(t))S_t dt + \sigma(t, S(t))S(t)dW(t)$$

then, under the assumption that the interest rate still is constant, the Q-dynamics are

$$dS(t) = rS(t)dt + \sigma(t, S(t))S(t)dW^Q(t).$$

Hence, from a valuation (pricing) point of view we are much more interested in $\sigma(t, S(t))$ than in $\alpha(t, S(t))$. A model of this kind, where we allow the volatility to depend on time and today's stock price, is known as a *local volatility model*, and the function $\sigma(\cdot, \cdot)$ is the *local volatility*.

Mathematical preliminaries

Local times

Central in the derivation of the Dupire formula is a family of stochastic processes known as *local times*. The presentation here is formal and we refer to more advanced texts for proofs and more precise results.

Theorem 1 *Let X be the 1-dimensional Ito diffusion*

$$dX(t) = \alpha(t, X(t))X(t)dt + \sigma(t, X(t))X(t)W(t).$$

There exists a family of stochastic processes

$$(L_t^x(X),\ t \geq 0,\ x \in \mathbb{R})$$

such that for every bounded measurable function f

$$\int_0^t f(X(u))\sigma^2(u, X(u))X^2(u)du = \int_{-\infty}^{\infty} f(x)L_t^x(X)dx. \tag{1}$$

For a fixed $x \in \mathbb{R}$ the family $(L_t^x(X))$ is called the *local time* at x of X. See Section 4.1 in Jeanblanc et al [2] for a proof of the theorem and more on local times

If we take the differential (with respect to t) of Equation (1) we get

$$f(X(t))\sigma^2(t, X(t))X^2(t)dt = \int_{-\infty}^{\infty} f(x)dL_t^x(X)dx,$$

and taking the expected value of this relation yields

$$E\left[f(X(t))\sigma^2(t, X(t))X^2(t)dt\right] = E\left[\int_{-\infty}^{\infty} f(x)dL_t^x(X)dx\right]. \tag{2}$$

Now

$$\text{LHS in (2)} = \int_{-\infty}^{\infty} f(x)\sigma^2(t, x)x^2 dt\, p(t, x)dx,$$

where $p(t, x)$ is the density function of $X(t)$, and

$$\text{RHS in (2)} = \int_{-\infty}^{\infty} f(x)E\left[dL_t^x(X)\right]dx.$$

Hence we have

$$\int_{-\infty}^{\infty} f(x)\left[\sigma^2(t, x)x^2 p(t, x)dt - E\left[dL_t^x(X)\right]\right]dx = 0$$

for every measurable and bounded f. It follows that the function within brackets must be zero, or

$$E\left[dL_t^x(X)\right] = \sigma^2(t, x)x^2 p(t, x)dt. \tag{3}$$

A generalized Ito formula

In order to derive the Dupire equation using stochastic calculus we need a version of Ito's formula for functions that are convex but not necessarily C^2. This version of the Ito formula, known as the Meyer-Ito formula, includes the local times in the second order integral, and says that if f is a convex function and X is an Ito diffusion then

$$f(X(t)) = f(X(0)) + \int_0^t f'(X(s))dX_s + \frac{1}{2}\int_{-\infty}^{\infty} L_t^y(X)d\mu(y)$$

where f' is the left derivative of f and μ is the second derivative of f in the distribution sense. (See Theorem 70 in Protter [3] for a precise result.)

For any $x \in \mathbb{R}$ the function

$$f(y) = \max(y - x, 0)$$

is convex (but not C^2) with left derivative

$$f'(y) = \mathbf{1}(y > x).$$

The second derivative is Dirac's delta function at x:

$$\mu(y) = \delta_x(y).$$

Using these expression for f' and μ in the Meyer-Ito formula shows that we can write

$$\max(X(t) - x, 0) = \max(X(0) - x, 0) + \int_0^t \mathbf{1}(X(s) > x) dX(s) + \frac{1}{2} L_t^x(X). \quad (4)$$

Dupire's formula

We let

$$c(T, K) = e^{-rT} E^Q \left[\max(S(T) - K, 0)\right] \quad (5)$$

be the price at time 0 of a European call option with strike price K and maturity T, where S has Q-dynamics given by

$$dS(t) = rS(t)dt + \sigma(t, S(t))S(t)dW^Q(t).$$

Here r is the constant interest rate and W^Q is a Q-Wiener process. We assume that there exists prices for call options of every maturity $T \geq 0$ and strike price $K \geq 0$. Dupire's formula connects the local volatility with prices of call options, and states that

$$\sigma(T, K) = \sqrt{2 \frac{\frac{\partial c}{\partial T}(T, K) + rK \frac{\partial c}{\partial K}(T, K)}{K^2 \frac{\partial^2 c}{\partial K^2}(T, K)}}.$$

Hence, given data in the form of prices of European call options we can derive the local volatility implict in these prices: we get the *local volatility surface*. Note that this is not the same thing as the Black-Scholes implied volatility.

Derivation of the formula

One way of deriving Dupire's formula is to go through the following steps.

I. Use Equation (4) and integration by parts to show that

$$\begin{aligned} e^{-rT} \max(S(T) - K, 0) &= \max(S(0) - K, 0) \\ &\quad - r \int_0^T e^{-ru} \max(S(u) - K, 0) du \\ &\quad + \int_0^T e^{-ru} \mathbf{1}(S(u) > K) dS(u) \\ &\quad + \frac{1}{2} \int_0^T e^{-ru} dL_u^K(S). \end{aligned}$$

II. Use the equation derived in I to show that

$$\begin{aligned} c(T,K) &= \max(S(0)-K,0) + rK\int_0^T e^{-ru}Q(S(u)>K)du \\ &+\frac{1}{2}\int_0^T e^{-ru}K^2\sigma^2(u,K)q(u,K)du, \end{aligned}$$

where $q(t,x)$ is the density function under Q of $S(t)$.

III. Use (5) to show that

$$\frac{\partial c}{\partial K}(T,K) = -e^{-rT}Q(S(T)>K)$$

and

$$\frac{\partial^2 c}{\partial K^2}(T,K) = e^{-rT}q(T,K),$$

and use this in the expression for $c(T,K)$ from II above to arrive at Dupire's formula.

References

[1] Björk, T. (2009), 'Arbitrage Theory in Continuous Time', 3rd Ed., *Oxford University Press*

[2] Jeanblanc, M., Yor, M. & Chesney M. (2009), 'Mathematical Methods for Financial Markets', *Springer-Verlag*

[3] Protter, P. E. (2004), 'Stochastic Integration and Differential Equations', 2nd Ed., *Springer-Verlag*