MTH 535 Fall 2004

A non-measurable set

Let X be a set. A relation $x \sim y$ is called an *equivalence relation* if for all $x, y, z \in X$,

i) $x \sim x$,

ii) $x \sim y \Longrightarrow y \sim z$,

iii) $x \sim y, \quad y \sim z \Longrightarrow x \sim z$.

For each $x \in X$ let $A_x := \{y \in X : y \sim x\}$ which we will call the *equivalence class* of x. Then for $u, v \in X$ either $A_u = A_v$ or $A_u \cap A_v = \emptyset$. Proof: Suppose $w \in A_u \cap A_v$. Then $w \sim u$ and $w \sim v$ so by ii) and iii) $u \sim v$. Thus if $z \in A_u$ we have $z \sim u$ and $u \sim v$ so by iii), $z \sim v \Longrightarrow z \in A_v$. Thus $A_u \subset A_v$ and similarly $A_v \subset A_u$ so $A_u = A_v$. Now also note that $x \in A_x$ by i). Thus each element of X belongs to exactly one equivalence class, and distinct equivalence classes are disjoint.

Let $X = (0, 1]$ and define a relation on X by $x \sim y$ iff $x - y$ is rational. It is easily checked that this is an equivalence relation. By the Axiom of Choice we can form a set S by selecting a single point from each equivalence class for this relation.

Theorem. The set S described above is not Lebesgue measurable.

If $x, y \in (0, 1]$ define $x \oplus y := x + y$ if $x + y \leq 1$, otherwise $x \oplus y := x + y - 1$. If $A \subset (0, 1]$ and $x \in (0, 1]$ we define $A \oplus x := \{a \oplus x : a \in A\}$.

We can show

Lemma. If A is measurable, so is $A \oplus x$ and $\mu_L(A \oplus x) = \mu_L(A)$.

We omit the proof.

Now we prove the theorem. Let $r_1, r_2, \ldots$ be an enumeration of the rationals in $(0, 1]$ (so each rational appears exactly once on the list.) We will show

1) If $i \neq j$ then $S \oplus r_i \cap S \oplus r_j = \emptyset$.

2) $(0, 1] = \cup_{i=1}^{\infty} S \oplus r_i$.

Proof of 1): Suppose $x \in S \oplus r_i \cap S \oplus r_j$. Then $x = s_i \oplus r_i = s_j \oplus r_j$ for some s_i, s_j. This implies that s_i and s_j differ by a rational, so $s_i = s_j$ since S contains exactly one member of each equivalence class. But then, since $0 < r_i, r_j \leq 1$ we would also have $r_i = r_j \Longrightarrow i = j$.

Proof of 2): If $x \in (0, 1]$ then $x \sim s$ for some $s \in S$ since x must be in some equivalence class, and a representative of each equivalence class appears in S. But then x differs from y by some rational number in $(0, 1]$ so that $x \in S \oplus r_i$ for some i.

Now we finish the proof. *If S were measurable*, with $\mu_L(S) = a$ then by the lemma we would have

$$1 = \mu_L((0,1]) = \sum_{i=1}^{\infty} \mu_L(S \oplus r_i) \quad (= a + a + a + \cdots)$$

so the sum on the right is either 0 or ∞ ; a contradiction in either case. Thus S is not measurable.