1 The multi period model

1.1 The model setup

In the multi period model time runs in discrete steps from $t = 0$ to $t = T$, where T is a fixed time horizon. As before we will assume that there are two assets on the market, a bond with price process B_t, and a stock with price process S_t

The bond price dynamics are given by

$$\begin{aligned} B_{n+1} &= (1+r)B_n, \\ B_0 &= 1, \end{aligned}$$

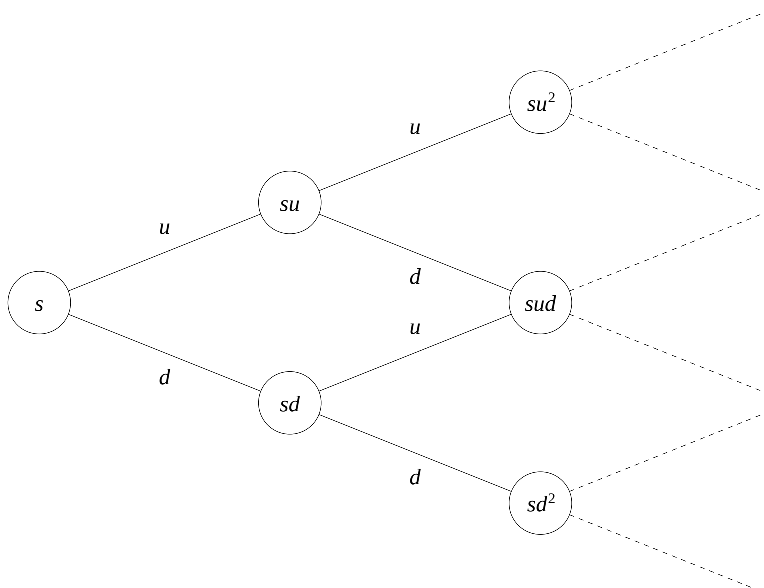
where r is the deterministic one period interest rate. The stock price dynamics are given by

$$\begin{aligned} S_{n+1} &= S_n Z_n, \\ S_0 &= s, \end{aligned}$$

where $Z_0, \ldots, Z_{T-1}$ are i.i.d. random variables such that

$$P(Z_n = u) = p, \qquad P(Z_n = d) = 1 - p.$$

The stock price dynamics can be illustrated with the following tree.



As before, our goal is to use the model for pricing and hedging of financial derivatives. In the multi period model the definition of a financial derivative, or contingent claim, is the following.

Definition 1 *A* financial derivative *or* contingent claim *is a random variable of the form*

$$X = \phi(S_T),$$

where the contract function ϕ *is a real valued function.*

We will start by considering a concrete example, and then we will make sure that we did the right thing afterwards.

Example 1 Suppose that the parameters of the model are $r = 0$, $u = 1.5$, $d = 0.5$, $p = 0.6$, and that we want to price a European call option with strike price K=80 and exercise time T=3. This means that we are looking at a claim X such that

$$X = \max\{S_T - K, 0\}.$$

Just like in the one period model it has to hold that

$$\Pi(T; X) = X,$$

or there will be arbitrage! So

$$\Pi(3; X) = \max\{S_3 - 80, 0\} = \begin{cases} 190 & \text{if } S_3 = 270, \\ 10 & \text{if } S_3 = 90, \\ 0 & \text{if } S_3 = 30, \\ 0 & \text{if } S_3 = 10. \end{cases}$$

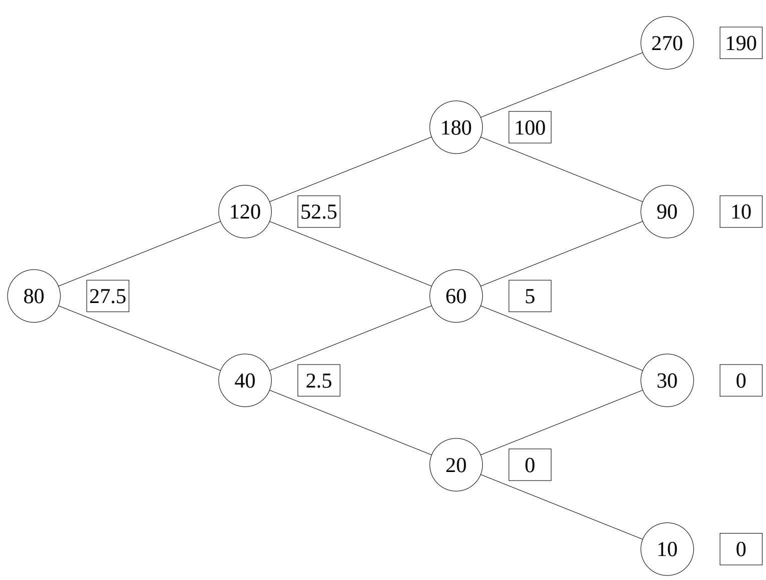
The problem is to find the price at time $t < T$, $\Pi(t, X)$. How do we go about doing this? Well, in the one period model it turned out that one should should use risk neutral valuation. What if we use the one period model results and work our way backwards in the tree one step at a time? We should use the martinagle probabilities which are obtained from

$$S(0) = \frac{1}{1+r} E^Q[S(1)].$$

or

$$s = \frac{1}{1+r}(q \cdot su + (1-q) \cdot sd).$$

Solving for q, and inserting our parameters we obtain $q = \frac{(1+r)-d}{u-d} = 0.5$. Working our way backwards through the tree we get that the price of the option at time $t = 0$ should be $\Pi(0, X) = 27.5$. See the figure below.



The question is now if we have really done the right thing? Have we found the arbitrage free price of the call option? Well, if we can find a replicating portfolio h, such that $V^h(0) = 27.5$

then the answer is yes. To find the replicating portfolio we start at $t = 0$ and work our way forward in the tree. To find the replication portfolio we need to find $h = (x, y)$ that solves

$$V^h(1) = \phi(Z_0).$$

This will result in

$$\begin{aligned} x &= \frac{1}{1+r} \cdot \frac{u\phi(d) - d\phi(u)}{u-d} \\ y &= \frac{1}{s} \cdot \frac{\phi(u) - \phi(d)}{u-d} \end{aligned}$$

For the first time step we have

$$\phi(Z_0) = \begin{cases} 52.5 & \text{if } Z_0 = u, \\ 2.5 & \text{if } Z_0 = d. \end{cases}$$

Solving for the replicating portfolio we find that $h = (-22.5, 5/8)$. The value of the portfolio at time $t = 0$ is

$$V^h(0) = -22.5 + \frac{5}{8} \cdot 80 = 27.5,$$

which is equal to the price of the option! So far, so good. Now if we hold the portfolio and $Z_0 = u$ the old portfolio will be worth

$$-22.5 + \frac{5}{8} \cdot 120 = 52.5$$

Solving $V^h(2) = \phi(Z_1)$, starting from $S_1 = 120$ will give us the portfolio $h = (-42.5, 95/120)$. The values of this new portfolio is

$$-42.5 + \frac{95}{120} \cdot 120 = 52.5$$

There is thus exactly enough money to buy the new portfolio once we have sold the old one. This is very important, and we say that the rebalancing of the portfolio *self-financing*.

Also note that the one period model is complete if the number of assets including the bond is equal to the number of outcomes in the sample space. For the multi period model we need intermediary trading or the model would not be complete.

Solving $V^h(2) = \phi(Z_1)$, starting from $S_1 = 40$ will give us the portfolio $h = (-2.5, 1/8)$. The old portfolio is now worth

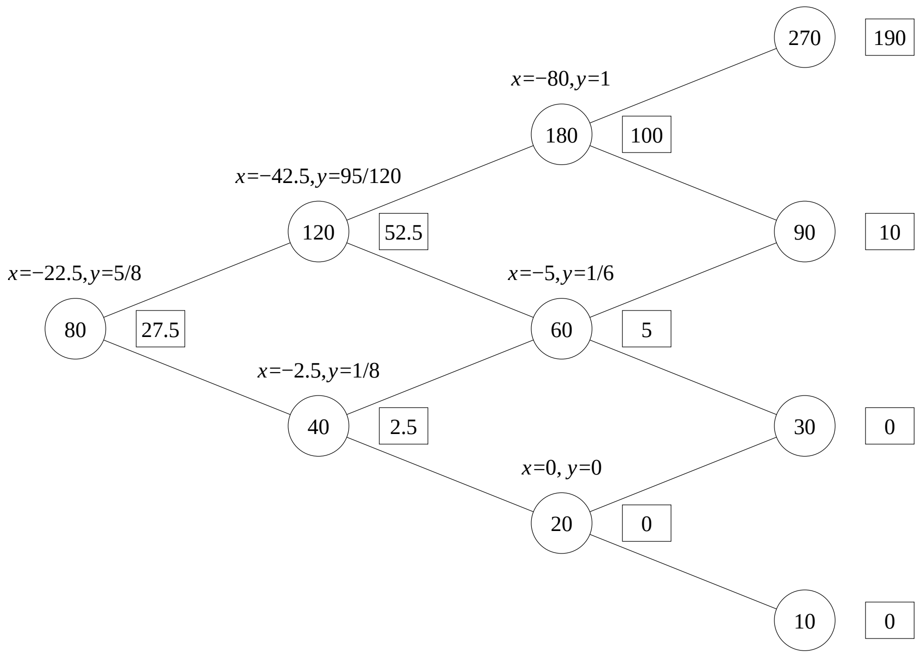
$$-22.5 + \frac{5}{8} \cdot 40 = 2.5$$

The new portfolio is worth

$$-2.5 + \frac{1}{8} \cdot 40 = 2.5$$

The rebalancing is self-financing!

We can now continue working our way forward in the tree and will end up with the following picture, and it is easy to check that portfoilo rebalancing is self-financing at every node.



We have therefore found the arbirtage free price of the option at time $t = 0$, $\Pi(0, X) = 27.5$, that is, since there is a self-financing portfolio replicating the payoff of the option with the same value at time $t = 0$. $\square$

1.2 Arbitrage and completeness

Now we do the more formal analysis of the multi period model. The biggest difference is that we need to indtroduce the concept of self-financing portfolios and make sure that the portfolios are predictable. Here are the definitions for the multi period model.

Definition 2 *A* portfolio strategy *is a stochastic process*

$$\{h_t = (x_t, y_t);\ t = 1, \ldots, T\}$$

such that $h_t \in \sigma\{S_0, \ldots, S_{t-1}\}$. *By convention* $h_0 = h_1$. *We have that*

$$\begin{aligned} x_t &= \textit{number of SEK in the bank } (t-1, t] \\ y_t &= \textit{number of stocks you own } (t-1, t] \end{aligned}$$

The value process V^h *corresponding to the portfolio strategy* h *is given by*

$$V_t^h = x_t(1 + r) + y_t S_t.$$

A portfolio strategy h *is* self-financing *if*

$$x_t(1 + r) y_t S_t = x_{t+1} + y_{t+1} S_t \quad t\text{=}1,\ldots,T\text{-}1$$

A portfolio strategy h *is said to be an* arbitrage portfolio *if* h *is self-financing and*

$$\begin{aligned} V^h(0) &= 0, \\ P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0. \end{aligned}$$

The claim $X = \phi(S_T)$ is reachable *if there exists a self-financing portfolio strategy h such that*

$$V_T^h = \phi(S_T) \quad \text{with probability 1.}$$

If such a portfolio h exists is said to be a hedging *or* replicating portfolio *for the claim X.*

From now on we will assume that $u < d$.

Proposition 1 *The model is free of arbitrage (i.e. there exists no arbitrage portfolios) if and only if*

$$d < (1 + r) < u.$$

The proof will be given at the end of Section 1.3. Just as for the one period model we will have use for the martingale measure Q.

Definition 3 *Q is a* martingale measure *if*

1. $0 < q < 1$ *(means $Q \sim P$)*
2. $s = \dfrac{1}{1+r} E^Q[S_{t+1}|S_t = s]$.

The expression for q will be the same as for the one period model.

Proposition 2 *The martingale probabilities are given by*

$$q = \frac{(1 + r) - d}{u - d},$$

and $1 - q$.

Proposition 3 *The multi period binomial model is complete (i.e. all claims are reachable).*

Proof: The one period model + induction (see Example 1). □

1.3 Pricing

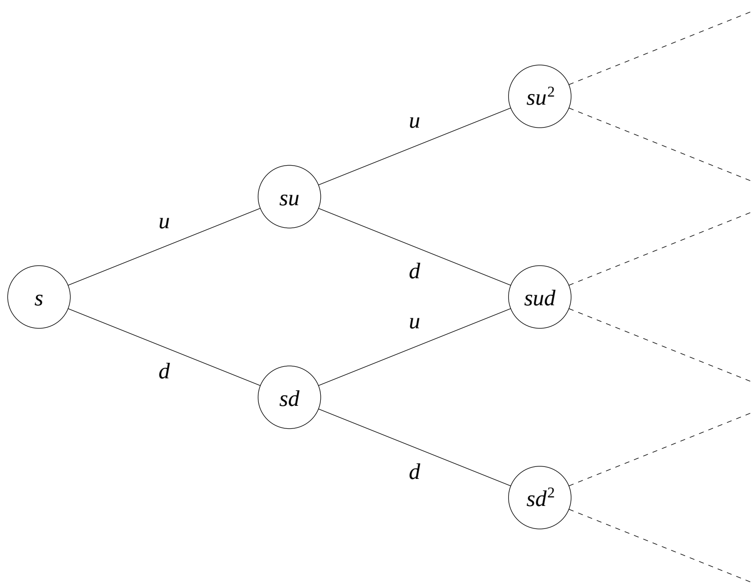
If $X = \phi(S_T)$ is replicated by the portfolio h then

$$\Pi(t; X) = V^h(t), \quad t = 0, \ldots, T.$$

Every node in the tree describing the stock price evolution is described by two indices (t, k) where

$$\begin{aligned} t &= \text{time} \\ k &= \text{the number of ``up-steps''} \end{aligned}$$

We have that $S_t(k) = su^k d^{t-k}$ where $k = 0, 1, \ldots, t$.



If we denote by $V_t(k)$ the value of the replicating portfolio h at the node (t,k), then $V_t(k)$ can be computed recursively according to

$$\begin{cases} V_{t-1}(k) &= \dfrac{1}{1+r}\{qV_t(k+1)+(1-q)V_t(k)\}, \\ V_T(k) &= \phi(su^k d^{T-k}), \end{cases}$$

where

$$q = \frac{(1+r)-d}{u-d}.$$

The replicating portfolio is given by

$$\begin{cases} x_t(k) &= \dfrac{1}{1+r}\dfrac{uV_t(k)-dV_t(k+1)}{u-d}, \\ y_t(k) &= \dfrac{1}{S_{t-1}}\dfrac{V_t(k+1)-V_t(k)}{u-d}. \end{cases}$$

From the above algorithm we can obtain a risk neutral valuation formula.

Proposition 4 *The arbitrage free price at $t=0$ of the claim X is given by*

$$\Pi(0;X) = V_0 = \frac{1}{(1+r)^T}E^Q[X]$$

where Q denotes the martingale measure, or more explicitly

$$\Pi(0;X) = \frac{1}{(1+r)^T}\sum_{k=0}^{T}\binom{T}{k} q^k(1-q)^{T-k}\phi(su^k d^{T-k})$$

Proof: To see that the second formula is correct let Y = number of "up-steps". Then $Y \in Bin(T,q)$. We obtain

$$X = \phi(S_T) = \phi(su^Y d^{T-Y})$$

Take expectation to obtain the explicit formula. □

Now we can prove Proposition 1, abscence of arbitrage.
Proof: That abscence of arbitrage implies that $d < (1+r) < u$ follows from the corresponding result for the one period model.
Suppose that $d < (1+r) < u$, then we need to prove abscence of arbitrage. For this purpose fix an arbitrary self-financing portfolio h such that

$$\begin{aligned} P(V_T^h \geq 0) &= 1, \\ P(V_T^h > 0) &> 0, \end{aligned}$$

h is then a potential arbitrage portfolio. From this and the risk neutral valuation formula we get that

$$V_0^h = \frac{1}{(1+r)^T} E^Q[V_T^h] > 0,$$

which means that h is not an arbitrage portfolio. □