1 Dividends

In this section we will study how to price financial derivatives written on dividend paying underlying assets. We shall to begin with only consider discrete dividend payments. First you need to know what happens to the stock price when a dividend is paid out.
Suppose that $S(t)$ denotes the price **ex dividend**, i.e. right after dividend payment, of the underlying asset at time t. If a dividend payment is made at time t, we will actually think of it as being made at time $t - dt$, or t_- if you will. Let the size of the dividend be d, then a (slightly heuristic) arbitrage argument gives the following result.

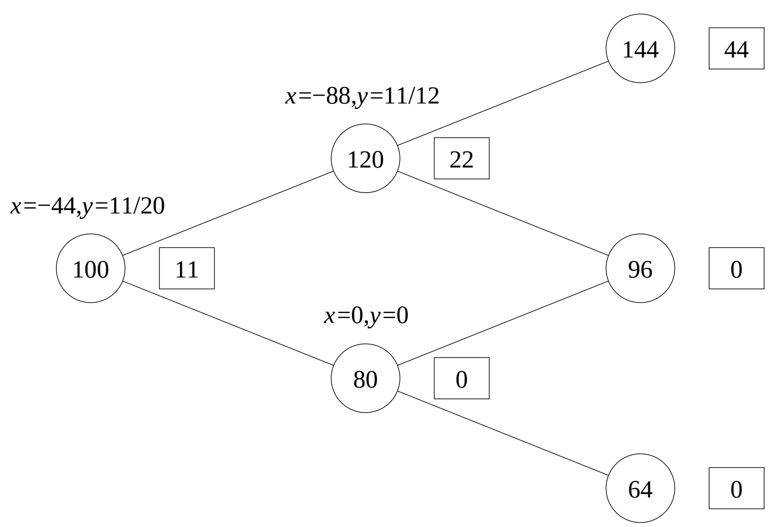
Proposition 1 (Jump condition) *In order to avoid arbitrage possibilities the following jump condition must hold at every dividend point t.*

$$S_t = S_{t-} - d$$

We will now consider an example to see what changes when the underlying asset pays dividends.

Example 1 Consider pricing a European call option with strike price $K = 100$, using a two period binomial model with the parameters $r = 0$, $u = 1.2$, $d = 0.8$, $p = 0.8$ and given that the stock price today is $S(0) = 100$.

- **Case with no dividends** Using what we know about the binomial model we will then get the result presented in the figure below.



In general when there are no dividends the self-financing condition for a portfolio is

$$h_t S_t = h_{t+1} S_t.$$

Introducing the notation $\Delta X_t = X_t - X_{t-1}$ this can be written as

$$S_t \Delta h_{t+1} = 0.$$

In general if $V_t = h_t S_t$ then

$$\Delta V_t = h_t S_t - h_{t-1} S_{t-1} = h_t S_t - h_t S_{t-1} + h_t S_{t-1} - h_{t-1} S_{t-1} = h_t \Delta S_t + S_{t-1} \Delta h_t.$$

This means that for a self-financing portfolio we have

$$V_t = V_0 + \sum_{n=1}^{t} \Delta V_n = V_0 + \sum_{n=1}^{t} h_n \Delta S_n.$$

For the concrete example above we have

$$22 = 11 + (-44) \cdot 0 + \frac{11}{20}(\underbrace{120 - 100}_{\Delta S})$$

Furthermore we know that the normalized price process

$$Z_t = \frac{S_t}{B_t}$$

is a Q-martingale. In the example this means that

$$S_0 = E^Q \left[\frac{S_1}{B_1} \right],$$

which yields

$$100 = \frac{1}{1+0} \left[q \cdot 120 + (1-q) \cdot 80 \right],$$

resulting in

$$q = \frac{1}{2}.$$

•**Case with dividends** Now let all the parameters be the same, but assume that at time $t = 1$ the stock pays a dividend of

$$d_1 = 0.1 \cdot S_{1-}$$

Recall the jump condition

$$S_1 = S_{1-} - d_1$$

Thus if the stock price goes up we will have

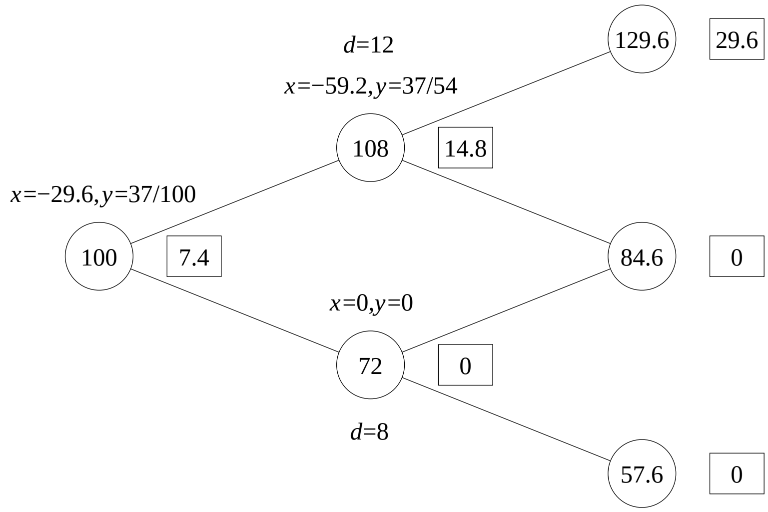
$$d_1 = 0.1 \cdot 1.2 \cdot 100 = 12,$$

and

$$S_1 = 1.2 \cdot 100 - 12 = 108.$$

How will this effect the price of the option? It will become cheaper! The reason is that the stock price at time $t = 0$ includes the value of the dividend payment, but it is the owner of the stock who receives the dividend payment, not the owner of the option.

The price of the call option with strike $K = 100$ has been computed and result is presented in the figure below.



Let us look at what has changed. Computing the portfolio at time $t = 0$ the following system should be solved

$$x + y(108 + 12) = -59.2 + \tfrac{37}{54} \cdot 108 = 14.8$$

$$x + y(72 + 8) = 0.$$

Note that you are allowed to finance the purchase of the new portfolio using the dividend payment!

In general when the underlying asset pays dividends a self-financing portfolio will have the following dynamics.

$$V_t = V_0 + \sum_{n=1}^{t} \Delta V_n = V_0 + \sum_{n=1}^{t} h_n \Delta G_n.$$

where the gain process G is defined by $G_t = S_t + D_t$. Here D is the cumulative dividend process.

In the example we have

$$14.8 = 7.4 + (-29.6) \cdot 0 + \frac{37}{100}(\underbrace{108 - 100}_{\Delta S} + \underbrace{12}_{\Delta D})$$

Furthermore the definition of a martingale measure will change. Instead of the normalized price process, the normalized gain process G^Z will be a Q-martingale. The normalized gain process G^Z is defined in the following way

$$G^Z_t = \frac{S_t}{B_t} + \sum_{n=1}^{t} \frac{1}{B_n} \Delta D_n$$

In the example we have

$$S_0 = E^Q \left[\frac{S_1}{B_1} + \frac{D_1 - D_0}{B_1} \right],$$

which yields

$$100 = \frac{1}{1+0} \left[q \cdot (108 + 12) + (1 - q) \cdot (72 + 8) \right]$$

resulting in

$$q = \frac{1}{2}.$$

$\square$

1.1 The model setup

Consider a discrete time model, so $t \in \{0, 1, \ldots, T\}$. Assume that there is a risk less asset with price process S^0 and a risky asset with price process S^1 paying dividends. Dividend payments are specified by the cumulative dividend process D in such a way that D^1_t is the total amount of dividend payments made over the interval $[0, t]$, more precisely $\Delta D^1_t = D^1_t - D^1_{t-1}$ is the dividend paid at time t (or t_-). The model is thus specified by

$$[S(t), D(t)] = \begin{bmatrix} S^0(t) & 0 \\ S^1(t) & D^1(t) \end{bmatrix}$$

1.2 Gain process, normalization and martingale measure

As we saw in Example 1 the concept of a gain process will come in handy in the presence of dividends. Also as before analyzing the model will be easier if you normalize the economy. The risk less asset S^0 will be used as numeraire and we will use the notation $S^0_t = B_t$. Here are the formal definitions.

Definition 1 *Given $[S(t), D(t)]$ the* **gain process** *G is defined by*

$$G(t) = S(t) + D(t)$$

The normalized economy, *or Z-economy $[Z(t), D^Z(t)]$, is defined by*

$$\begin{cases} Z(t) &= \dfrac{S(t)}{B(t)}, \\ D^Z(t) &= \displaystyle\sum_{n=1}^{t} \frac{1}{B(n)} \Delta D(n) \end{cases}$$

i.e.

$$\Delta D^Z(t) = \frac{1}{B(t)} \Delta D(t)$$

*The Z-*gain process *G^Z is defined by*

$$G^Z(t) = Z(t) + D^Z(t)$$

In Example 1 we saw that the definition of a self-financing portfolio has to be changed if the underlying assets pay dividends.

Definition 2 *A* portfolio strategy *is a process $h = (h^0, h^1)$ which is predictable. The S-*value process *$V^S(h)$ is given by*

$$V^S_t(h) = \sum_{i=0}^{1} h^i_t S^i_t = h_t S_t.$$

The Z-value process $V^Z(h)$ *is given by*

$$V_t^Z(h) = \sum_{i=0}^{1} h_t^i Z_t^i = h_t Z_t.$$

A portfolio is said to be S-self-financing *if*

$$V_t^S(h) = V_0^S(h) + \sum_{n=1}^{t} h_n \Delta G_n$$

and Z-self-financing *if*

$$V_t^Z(h) = V_0^Z(h) + \sum_{n=1}^{t} h_n \Delta G_n^Z$$

Remark 1 Note that the definition of value process does nor change, the dividends are included in the prices process of the stock.

Finally, the definition of a martingale measure has to be changed if there are dividend-paying assets on the market. What should be a martingale under the martingale measure is the normalized gain process.

Definition 3 *A probability measure* Q *is said to be a* martingale measure *for the market* $[S, D]$ *if*

- $Q \sim P$,
- *The* Z*-gain process* G^Z *is a* Q*-martingale.*

1.3 Pricing

Suppose that the price process of the risk less asset B is given by $B(t) = e^{rt}$, where r is a constant. This means that the risk less asset pays a continuously compounded interest of r. Fix a martingale measure Q, should there be several.
Fix a contingent T-claim X. We want the $[S, D]$-market adjoined by $[\Pi(X), 0]$ to be free of arbitrage. This will be true if all normalized gain processes are Q-martingales and this will be true if the price of X at time t is given by

$$\begin{aligned} \Pi_t[X; Q] &= B_t E^Q \left[\left. \frac{X}{B_T} \right| \mathcal{F}_t \right] \\ &= e^{-r(T-t)} E^Q \left[X \middle| \mathcal{F}_t \right]. \end{aligned}$$

From the definition of a martingale measure we get the following.

Proposition 2 *For the pair* $[S^1, D^1]$ *it holds that*

$$S_t^1 = E^Q \left[e^{-r(T-t)} S_T^1 + \sum_{n=t+1}^{T} e^{-r(n-t)} \Delta D_n^1 \middle| \mathcal{F}_t \right].$$

In particular it holds that

$$S_0^1 = E^Q \left[e^{-rT} S_T^1 + \sum_{n=1}^{T} e^{-rn} \Delta D_n^1 \middle| \mathcal{F}_t \right].$$

This is known as the "cost-of-carry-formula": today's price is the discounted expectation of future earnings.
Proof: We know that G^Z is a Q-martingale. Recall that G^Z is

$$G^Z(t) = Z(t) + D^Z(t),$$

where

$$\begin{aligned} Z(t) &= \frac{S(t)}{B(t)}, \\ D^Z(t) &= \sum_{n=1}^{t} \frac{1}{B(n)} \Delta D(n) \end{aligned}$$

Thus

$$(G^Z)^1_t = \frac{S^1(t)}{B(t)} + \sum_{n=1}^{t} \frac{1}{B(n)} \Delta D^1(n)$$

Using the martingale property of $(G^Z)^1_t$

$$(G^Z)^1_t = E^Q[(G^Z)^1_T | \mathcal{F}_t]$$

we get

$$\frac{S^1(t)}{B(t)} + \sum_{n=1}^{t} \frac{1}{B(n)} \Delta D^1(n) = E^Q\left[\frac{S^1(T)}{B(T)} + \sum_{n=1}^{T} \frac{1}{B(n)} \Delta D^1(n) \middle| \mathcal{F}_t \right].$$

Since $\sum_{n=1}^{t} \frac{1}{B(n)} \Delta D(n) \in \mathcal{F}_t$ we can move it into the expectation, and since $B(t)$ is deterministic it can also be multiplied into the expectation. We obtain

$$S^1(t) = E^Q\left[B(t) \frac{S^1(T)}{B(T)} + \sum_{n=t+1}^{T} \frac{B(t)}{B(n)} \Delta D^1(n) \middle| \mathcal{F}_t \right]$$

Finally use that $B(t) = e^{rt}$ and the result will follow. □

1.4 Dividends in the binomial model

There are two cases. The first case is that of known dividend yield paid out discretely in time. That is what we looked at in Example 1. The second case is that of a known dollar (or other currency) dividend.

1.4.1 Known dividend yield

As mentioned, this is what we looked at in Example 1 and the dividend payment is then of the form $\Delta D_t = \delta S_{t_-}$, where $0 < \delta < 1$. Note that if we just want to compute the price of the option we can compute the parameters u, d, and q as if no dividends were paid and just adjust the stock price dynamics (for the replicating portfolio this will not work).

1.4.2 Known dollar dividend

Here we assume that $\Delta D_t = d$ for some fixed number d. To adjust the tree for this kind of dividend you could make sure to have one time step end when the dividend is paid out and have the stock price drop by the same amount. The problem with this approach is that the tree will no longer recombine since now

$$S_{t+1} = (S_t - D)Z$$

where Z is u with probability q and d with probability $1-q$. The term DZ will increase the number of nodes at time $t+1$.
An engineering fix for this problem is the following: decompose the stock price into a risky part S^* and a risk less part which is the present value of future dividends $PV(div)$

$$S_t = S_t^* + PV(div).$$

Then build a tree for S^* computing u, d and q using σ for S (actually σ^* should be used, which is slightly greater, than σ). Note that $S_0^* = S_0 - PV(div)$. Then compute the present value of future dividends at every node and add the to values to get the tree for stock price S. Now continue computing prices using q.

Example 2 Consider pricing a European call option with strike price $K = 110$ and maturity in one year, using a two period binomial tree. The current value of the underlying stock is $S_0 = 100$ and the volatility is σ=20%. The underlying asset will pay a dividend of \$13 in six months. The risk free interest rate with continuous compounding is 7.6%.

We have that

$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} \approx 1.1519 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.8681 \end{aligned}$$

The tree for S^* will look as follows

$$\begin{array}{ccccc}
 & & & & 116.08 \\
 & & & \nearrow & \\
 & & 100.77 & & \\
 & \nearrow & & \searrow & \\
87.48 & & & & 87.48 \\
 & \searrow & & \nearrow & \\
 & & 75.95 & & \\
 & & & \searrow & \\
 & & & & 65.93
\end{array}$$

The tree for the present value of future dividends $PV(div)$ will be

$$\begin{array}{ccccc}
 & & & & 0 \\
 & & & \nearrow & \\
 & & 0 & & \\
 & \nearrow & & \searrow & \\
12.52 & & & & 0 \\
 & \searrow & & \nearrow & \\
 & & 0 & & \\
 & & & \searrow & \\
 & & & & 0
\end{array}$$

Thus the tree for the stock price will look like

$$
\begin{array}{ccccc}
 & & & & 116.08 \\
 & & & \nearrow & \\
 & & 100.77 & & \\
 & \nearrow & & \searrow & \\
100 & & & & 87.48 \\
 & \searrow & & \nearrow & \\
 & & 75.95 & & \\
 & & & \searrow & \\
 & & & & 65.93
\end{array}
$$

Now the option price tree can be computed using

$$q = \frac{e^{0.076 \cdot 0.5} - d}{u - d} \approx 0.6012$$

and the discount factor

$$\frac{1}{e^{r \Delta t}} \approx 0.9627.$$

and the result is

$$
\begin{array}{ccccc}
 & & & & 6.08 \\
 & & & \nearrow & \\
 & & 3.52 & & \\
 & \nearrow & & \searrow & \\
2.04 & & & & 0.00 \\
 & \searrow & & \nearrow & \\
 & & 0.00 & & \\
 & & & \searrow & \\
 & & & & 0.00
\end{array}
$$

□