KTH Mathematics

Homework 1 in SF2701 Financial Mathematics, basic course, spring 2017.

Answers and some hints for solutions.

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1. (a) We have that

$$\begin{aligned} T &= 6/12 = 1/2 \\ \Delta t &= T/3 = 1/6 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.0851 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.9216 \end{aligned}$$

and the tree for the stock price is therefore

$$\begin{array}{cccc} & & & 127.7556 \\ & & 117.7389 & \\ & 108.5076 & & 108.5076 \\ 100.0000 & & 100.0000 & \\ & 92.1595 & & 92.1595 \\ & & 84.9337 & \\ & & & 78.2744 \end{array}$$

Now the option price tree can be computed using

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.5103$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0050}$$

and the result is

$$\begin{array}{cccc} & & & 27.7556 \\ & & 18.2377 & \\ & 11.3644 & & 8.5076 \\ 6.8385 & & 4.3194 & \\ & 2.1930 & & 0.0000 \\ & & 0.0000 & \\ & & & 0.0000 \end{array}$$

The price of the option is thus 6.8385.

For the replicating portfolio we obtain

$$
\begin{array}{ccccc}
 & & & & \begin{aligned} x &= -99.5012 \\ y &= 1.0000 \end{aligned} \\
 & & \begin{aligned} x &= -73.7725 \\ y &= 0.7846 \end{aligned} & & \\
\begin{aligned} x &= -49.2619 \\ y &= 0.5610 \end{aligned} & & & & \begin{aligned} x &= -47.7207 \\ y &= 0.5204 \end{aligned} \\
 & & \begin{aligned} x &= -24.2285 \\ y &= 0.2867 \end{aligned} & & \\
 & & & & \begin{aligned} x &= 0.0000 \\ y &= 0.0000 \end{aligned}
\end{array}
$$

Here x is the amount of money in the bank and y is the number of stock you own. Note that $y = \Delta$, and that once you have y it is easy to solve for x.

(b) Here we get the same price as for the option in exercise 1a, since we have an American call option on a non-dividend paying stock. The price of the option is thus 6.8385. Also the replicating portfolio is the same.

(c) The option price tree now becomes

$$
\begin{array}{cccc}
 & & & 0.0000 \\
 & & 0.0000 & \\
 & 1.8618 & & 0.0000 \\
5.4682 & & 3.8207 & \\
 & 9.2816 & & 7.8405 \\
 & & 15.0663 & \\
 & & & 21.7256
\end{array}
$$

In each node the value is obtained as

$$\max\{100 - S_t, \frac{1}{1.0050}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_t is the current stock price, and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 15.0663. The price of the option is thus 5.4682.

For the replicating portfolio we obtain

$$
\begin{array}{ccccc}
 & & & & \begin{aligned} x &= 0.0000 \\ y &= 0.0000 \end{aligned} \\
 & & \begin{aligned} x &= 25.2325 \\ y &= -0.2154 \end{aligned} & & \\
\begin{aligned} x &= 50.8544 \\ y &= -0.4539 \end{aligned} & & & & \begin{aligned} x &= 51.7806 \\ y &= -0.4796 \end{aligned} \\
 & & \begin{aligned} x &= 78.0703 \\ y &= -0.7464 \end{aligned} & & \\
 & & & & \begin{aligned} x &= - \\ y &= - \end{aligned}
\end{array}
$$

Again, x is the amount of money in the bank and y is the number of stock you own. Note that once the option has been exercised you can no longer replicate.

2. (a) Again we have that

$$
\begin{aligned}
T &= 6/12 = 1/2 \\
\Delta t &= T/3 = 1/6 \\
u &= e^{\sigma\sqrt{\Delta t}} \approx 1.0851 \\
d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.9216
\end{aligned}
$$

and the tree for the stock price is therefore

$$
\begin{array}{cccc}
 & & & 121.3678 \\
 & & 111.8520 & \\
 & 108.5076 & & 103.0822 \\
100.0000 & & 95.0000 & \\
 & 92.1595 & & 87.5515 \\
 & & 80.6870 & \\
 & & & 74.3607
\end{array}
$$

When the dividend is paid out the jump condition $S_t = S_{t-} - \delta S_{t-}$ is used. Now the option price tree can be computed using

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.5103$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0050}$$

and the result is

$$
\begin{array}{cccc}
 & & & 21.3678 \\
 & & 17.7389 & \\
 & 9.7689 & & 3.0822 \\
5.3470 & & 1.5649 & \\
 & 0.7945 & & 0.0000 \\
 & & 0.0000 & \\
 & & & 0.0000
\end{array}
$$

In each node the value is obtained as

$$\max\{S_{t-} - 100, \frac{1}{1.0050}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_{t-} is the current stock price **just before dividend payment**, and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 17.7389. The price of the option is thus 5.3470.

(b) We still have that

$$
\begin{aligned}
T &= 6/12 = 1/2 \\
\Delta t &= T/3 = 1/6 \\
u &= e^{\sigma\sqrt{\Delta t}} \approx 1.0851 \\
d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.9216
\end{aligned}
$$

but now we make the tree for S^* which starts at

$$S_0^* = S_0 - PV_0(div) = 100 - e^{-003 \cdot 4/12} 5 = 95.0498.$$

The tree for S^* is therefore

$$\begin{array}{cccc}
 & & & 121.4314 \\
 & & 111.9105 & \\
 & 103.1362 & & 103.1362 \\
95.0498 & & 95.0498 & \\
 & 87.5974 & & 87.5974 \\
 & & 80.7293 & \\
 & & & 74.3997
\end{array}$$

Now the stock price tree can be computed using $S_t = S_t^* + PV_t(div)$. The stock price tree becomes

$$\begin{array}{cccc}
 & & & 121.4314 \\
 & & 111.9105 & \\
 & 108.1112 & & 103.1362 \\
100.0000 & & 95.0498 & \\
 & 95.5724 & & 87.5974 \\
 & & 80.7293 & \\
 & & & 74.3997
\end{array}$$

The option price tree can be computed using

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.5103$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0050}$$

and the result is

$$\begin{array}{cccc}
 & & & 21.4314 \\
 & & 16.9105 & \\
 & 9.3616 & & 3.1362 \\
5.1470 & & 1.5923 & \\
 & 0.8084 & & 0.0000 \\
 & & 0.0000 & \\
 & & & 0.0000
\end{array}$$

In each node the value is obtained as

$$\max\{S_{t-} - 100, \frac{1}{1.0050}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_{t-} is the current stock price **just before dividend payment**, and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 16.9105. The price of the option is thus 5.1470.

3. (a) We have that the forward price is given by

$$f(0,T) = \frac{\Pi_0(S_T)}{p(0,T)} = e^{rT} E^Q \left[\frac{S_T}{B_T}\right].$$

This forward price will make the initial value of the forward contract zero.

To compute the expectation note that we have the formula

$$S_0 = E^Q\left[\frac{S_T}{B_T} + \sum_{t_i \le T} \frac{\Delta D_{t_i}}{B_{t_i}}\right]$$

Here we can insert any T and therefore we have that

$$S_0 = E^Q\left[\frac{S_{0.5}}{B_{0.5}} + \frac{\Delta D_{2/12}}{B_{2/12}} + \frac{\Delta D_{4/12}}{B_{4/12}}\right] \tag{1}$$

but also

$$S_0 = E^Q\left[\frac{S_{4/12-}}{B_{4/12-}} + \frac{\Delta D_{2/12}}{B_{2/12}}\right] \tag{2}$$

and

$$S_0 = E^Q\left[\frac{S_{2/12-}}{B_{2/12-}}\right]. \tag{3}$$

If we use that $\Delta D_{2/12} = 0.05 S_{2/12-}$ and insert this and (3) into (2), we obtain

$$S_0 = E^Q\left[\frac{S_{4/12-}}{B_{4/12-}} + \frac{0.05 S_{2/12-}}{B_{2/12-}}\right] = E^Q\left[\frac{S_{4/12-}}{B_{4/12-}}\right] + 0.05 S_0$$

or

$$E^Q\left[\frac{S_{4/12-}}{B_{4/12-}}\right] = (1-0.05)S_0$$

Inserting this, that $\Delta D_{4/12} = 0.05 S_{4/12-}$ and $\Delta D_{2/12} = 0.05 S_{2/12-}$, and (3) into (3a) we obtain

$$S_0 = E^Q\left[\frac{S_{0.5}}{B_{0.5}}\right] + 0.05(1-0.05)S_0 + 0.05 S_0$$

or

$$E^Q\left[\frac{S_{0.5}}{B_{0.5}}\right] = [1 - 0.05 - 0.05(1-0.05)]S_0 = (1-0.05)^2 S_0.$$

The forward price is therefore

$$f(0, 0.5) = e^{0.03 \cdot 0.5}(1-0.05)^2 100 \approx 91.61.$$

(b) The forward price of the new contract is still given by

$$f(0,T) = \frac{\Pi_0(S_T)}{p(0,T)} = e^{rT} E^Q\left[\frac{S_T}{B_T}\right].$$

Proceeding in the same way as in 3 we obtain (we move time zero to three months later and the second of the two dividend payments which were planned six months ago is now due in one month)

$$S_0 = E^Q\left[\frac{S_{0.5}}{B_{0.5}} + \frac{\Delta D_{1/12}}{B_{1/12}}\right]$$

and

$$S_0 = E^Q\left[\frac{S_{1/12-}}{B_{1/12-}}\right].$$

Using these equations, and that $\Delta D_{1/12} = 0.05 S_{1/12-}$ we find that

$$f(0, 0.5) = e^{0.03 \cdot 0.5}(1-0.05)95 \approx 91.61.$$

The price of the forward contract contracted three months ago is given by

$$\Pi_0(S_{3/12} - f_0) = \Pi_0(S_{3/12}) - p(0, 3/12)f_0 = \Pi_0(S_{3/12}) - e^{-0.03 \cdot 3/12} f_0$$

where $f_0 = 91.61$ is the forward price determined three months ago. To compute $\Pi_0(S_{3/12}) = E^Q[S_{3/12}/B_{3/12}]$ use

$$S_0 = E^Q\left[\frac{S_{3/12}}{B_{3/12}} + \frac{\Delta D_{1/12}}{B_{1/12}}\right]$$

and

$$S_0 = E^Q\left[\frac{S_{1/12-}}{B_{1/12-}}\right]$$

and that $\Delta D_{1/12} = 0.05 S_{1/12-}$. We obtain

$$E^Q\left[\frac{S_{3/12}}{B_{3/12}}\right] = (1 - 0.05)S_0.$$

Therefore

$$\Pi_0(S_{3/12} - f_0) = \Pi_0(S_{3/12}) - e^{-0.03 \cdot 3/12} f_0 = (1 - 0.05)95 - e^{-0.03 \cdot 3/12} 91.61 = -0.6794$$

4. For the futures price tree we have the following parameters

$$\begin{aligned} T &= 6/12 = 1/2 \\ \Delta t &= T/3 = 1/6 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.0851 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.9216 \end{aligned}$$

and the tree for the futures prices is therefore

			127.7556
		117.7389	
	108.5076		108.5076
100.0000		100.0000	
	92.1595		92.1595
		84.9337	
			78.2744

Now the option price tree can be computed using

$$q = \frac{1 - d}{u - d} \approx 0.4796$$

note the difference in q as compared with when it is computed from a stock price tree, and the discount factor (which is the same as before)

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0050}$$

and the result is

			27.7556
		17.7389	
	10.5673		8.5076
6.0460		4.0599	
	1.9374		0.0000
		0.0000	
			0.0000

In each node the value is obtained as

$$\max\{S_t - 100, \frac{1}{1.0050}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_t is the current stock price, and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 17.7389. The price of the option is thus 6.0460.

Note that you get early exercise for all American call options in this exercise set, except for the case when you have a non-dividend paying underlying (in which case you know that early exercise is never optimal).