KTH Mathematics

Homework 2 in SF2701 Financial Mathematics, basic course, spring 2017.

Answers and solutions.

1. (a) Use the Black-Scholes formula

$$c(0,s) = sN[d_1(0,s)] - e^{-rT}KN[d_2(0,s)].$$

where

$$\begin{cases} d_1(0,s) &= \frac{1}{\sigma\sqrt{T}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right\}, \\ d_2(0,s) &= d_1 - \sigma\sqrt{T}. \end{cases}$$

and N denotes the cumulative distribution function of the standard normal distribution, with parameters

$$s = 100, \quad K = 100, \quad \sigma = 0.2, \quad r = 0.03, \quad T = 0.5$$

to obtain

$$c = 6.3710.$$

(b) Use the Black-Scholes formula with parameters

$$s = S_0 - PV(div) = 100 - 5e^{-0.03\cdot 4/12}, \quad K = 100, \quad \sigma = 0.2, \quad r = 0.03, \quad T = 0.5$$

to obtain

$$c = 3.8983.$$

Note that this is the type of dividend which is not consistent with lognormal spot prices.

(c) Use the Black-Scholes formula with parameters

$$s = S_0 e^{-\delta T} = 100e^{-0.05\cdot 0.5}, \quad K = 100, \quad \sigma = 0.2, \quad r = 0.03, \quad T = 0.5$$

to obtain

$$c = 5.0493.$$

2. (a) Use the Black -76 formula (which can be obtained from Black-Scholes formula using $s = e^{-r(T-t)}F_t$) with parameters

$$F_0 = 100, \quad K = 100, \quad \sigma = 0.2, \quad ,r = 0.03, \quad T = 0.5$$

or the Black-Scholes formula with parameters

$$s = F_0 e^{-rT} = 100e^{-0.03\cdot 0.5}, \quad K = 100, \quad \sigma = 0.2, \quad r = 0.03, \quad T = 0.5.$$

This yields

$$c_{fut}(0) = 5.5533.$$

(b) Put-call parity for futures options is

$$p_{fut}(0) = e^{-rT}K - e^{-rT}F_0 + c_{fut}(0).$$

This obtained from the standard Black-Scholes put-call parity by substituting $s = e^{-rT}F_0$ everywhere.

For at the money futures options we see that the put and call price are equal, and thus

$$p_{fut}(0) = 5.5533.$$

3. The value of the total portfolio is given in the last row of the table below. From this we see that you will have gained 5.47 from selling the option and then delta hedging it.

time t	0	1/3	2/3	1
Stock price	100.00	114.00	104.00	108.00
Option price	6.78	12.87	3.81	0.00
Δ of option	0.4861	0.7313	0.4504	-
Value of option position	-6.78	-12.87	-3.81	0.00
Value of stock position	48.61	83.37	46.84	48.64
$d\Delta$	0.4861	0.2452	-0.2809	-
Bank	-41.83	-70.49	-42.46	-43.17
Value of total portfolio	0.00	0.01	0.57	5.47

Below you will find explanations of how the values have been computed. The notation $\Delta t = 1/3$ is used.

Value of option position $= (-1)\cdot$ option price,
Value of stock position $= \Delta\cdot$ stock price,
$d\Delta(t) = \Delta(t) - \Delta(t - \Delta t)$ (i.e. the change in the delta of the option),
Bank(t) = Bank$(t - \Delta t)\cdot e^{r\Delta t} - d\Delta\cdot$ stock price (stock price at t),
Value of total portfolio = Value of option position + Value of stock position + Bank.

4. Denote the payoff function by ϕ and note that

$$\phi(S_T) = -a + \max\{S_T - x_1, 0\} - 2\max\{S_T - x_2, 0\} + \max\{S_T - x_3, 0\}.$$

The price of the contract is therefore

$$\begin{aligned}
\Pi_t &= e^{-r(T-t)}E^Q[-a + \max\{S_T - x_1, 0\} - 2\max\{S_T - x_2, 0\} \\
&\quad + \max\{S_T - x_3, 0\}|\mathcal{F}_t] \\
&= -e^{-r(T-t)}a + c(t, S_t, x_1, T, r, \sigma) - 2c(t, S_t, x_2, T, r, \sigma) \\
&\quad + c(t, S_t, x_3, T, r, \sigma) \\
&= -e^{-r(T-t)}a + c(t, S_t, 0.95S_t, T, r, \sigma) - 2c(t, S_t, S_t, T, r, \sigma) \\
&\quad + c(t, S_t, 1.05S_t, T, r, \sigma).
\end{aligned}$$

Here $c(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ. If we want the price of the claim to be zero a should be chosen as

$$a = e^{r(T-t)} \left[c(t, S_t, 0.95S_t, T, r, \sigma) - 2c(t, S_t, S_t, T, r, \sigma) + c(t, S_t, 1.05S_t, T, r, \sigma) \right],$$

where $c(t, s, K, T, r, \sigma)$ is given by the Black-Scholes formula.