KTH Mathematics

Homework 2 in SF2701 Financial Mathematics, basic course, spring 2018.

Answers and solutions.

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1. (a) Use the Black-Scholes formula

$$c(0,s) = sN[d_1(0,s)] - e^{-rT}KN[d_2(0,s)].$$

where

$$\begin{cases} d_1(0,s) &=& \frac{1}{\sigma\sqrt{T}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right\}, \\ d_2(0,s) &=& d_1 - \sigma\sqrt{T}. \end{cases}$$

and N denotes the cumulative distribution function of the standard normal distribution, with parameters

$$s = 100, \quad K = 95, \quad \sigma = 0.3, \quad r = 0.02, \quad T = 0.75$$

to obtain

$$c = 13.5334.$$

(b) Use the Black-Scholes formula with parameters

$$s = S_0 - PV(div) = 100 - 5e^{-0.02 \cdot 6/12}, \quad K = 95, \quad \sigma = 0.3, \quad r = 0.02, \quad T = 0.75$$

to obtain

$$c = 10.4978.$$

Note that this is the type of dividend which is not consistent with lognormal spot prices.

(c) Use the Black-Scholes formula with parameters

$$s = S_0e^{-\delta T} = 100e^{-0.05 \cdot 0.75}, \quad K = 95, \quad \sigma = 0.3, \quad r = 0.02, \quad T = 0.75$$

to obtain

$$c = 12.1234.$$

2. (a) Use the Black -76 formula (which can be obtained from Black-Scholes formula using $s = e^{-r(T-t)}F_t$) with parameters

$$F_0 = 1215, \quad K = 1210, \quad \sigma = 0.165, \quad ,r = 0.03, \quad T = 0.5$$

or the Black-Scholes formula with parameters

$$s = F_0 e^{-rT} = 1215e^{-0.03\cdot 0.5}, \quad K = 1210, \quad \sigma = 0.165, \quad r = 0.03, \quad T = 0.5.$$

This yields

$$c_{fut}(0) = 58.0623.$$

(b) Put-call parity for futures options is

$$p_{fut}(0) = e^{-rT}K - e^{-rT}F_0 + c_{fut}(0).$$

This obtained from the standard Black-Scholes put-call parity by substituting $s = e^{-rT}F_0$ everywhere.

For at the money futures options we see that the put and call price are equal, and thus

$$p_{fut}(0) = 53.1367.$$

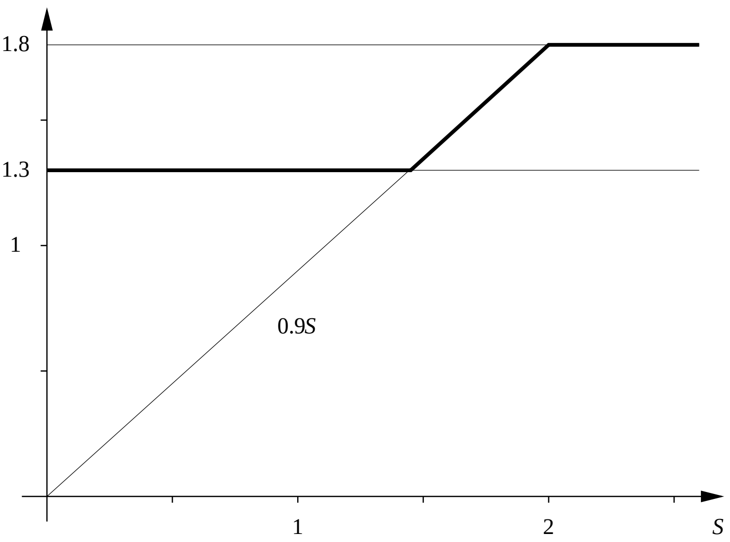
3. The value of the total portfolio is given in the last row of the table below. From this we see that you will have gained 9.18 from selling the option and then delta hedging it.

time t	0	1/3	2/3	1
Stock price	100.00	114.00	104.00	108.00
Option price	11.1117	17.0142	7.1982	3.0000
Δ of option	0.5348	0.7054	0.5355	-
Value of option position	-11.1117	-17.0142	-7.1982	-3.0000
Value of stock position	53.4810	80.4103	55.6914	57.8334
$d\Delta$	0.5348	0.1705	-0.1699	-
Bank	-42.3693	-62.2371	-45.1972	-45.6515
Value of total portfolio	0.00	1.1590	3.2960	9.1819

Below you will find explanations of how the values have been computed. The notation $\Delta t = 1/3$ is used.

Value of option position $= (-1)\cdot$ option price,
Value of stock position $= \Delta\cdot$ stock price,
$d\Delta(t) = \Delta(t) - \Delta(t - \Delta t)$ (i.e. the change in the delta of the option),
Bank(t) = Bank$(t - \Delta t)\cdot e^{r\Delta t} - d\Delta\cdot$ stock price (stock price at t),
Value of total portfolio = Value of option position + Value of stock position + Bank.

4. (a) The contract function of the claim is given by the bold curve in the figure below.



We see that the contract function is piecewise linear and therefore we know that it can be replicated using a constant portfolio consisting of bonds, call options, and the underlying stock. For the particular claim in this exercise we have

$$\begin{aligned} X &= \min\{\max\{1.3, 0.9\widetilde{S}_T\}, 1.8\} = \\ &= 1.3 + 0.9\left[\max\left\{\widetilde{S}_T - \frac{1.3}{0.9}, 0\right\} - \max\{\widetilde{S}_T - 2, 0\}\right] \end{aligned}$$

The replicating portfolio thus consists of 1.3 bonds with maturity T, 0.9 call options with time of maturity T and strike price 1.3/0.9, and a short position of 0.9 call options with time of maturity T and strike price 2. The price of the claim is therefore given by

$$\Pi(0; X) = 1.3e^{-rT} + 0.9C_\delta^{1.444}(0, 1) - 0.9C_\delta^2(0, 1),$$

where $C_\delta^K(t, s)$ denotes the price at time t of a European call option with time of maturity T and strike price K, when the value of the underlying at time t is s and the underlying has a dividend yield δ and volatility σ, and the interest rate is r.

Now we have that

$$\Pi(0; X) = 1.3e^{-rT} + 0.9C^{1.444}(0, 1 \cdot e^{-\delta(T-t)}) - 0.9C^2(0, 1 \cdot e^{-\delta(T-t)}),$$

where $C^K(t, s)$ denotes the price at time t of a European call option with time of maturity T and strike price K, when the value of the underlying at time t is s and the underlying has **no** dividend yield, volatility σ, and the interest rate is r. Thus $C^K(t, s)$ is given by the Black-Scholes formula and we obtain

$$\Pi(0; X) = 1.3e^{-rT} + 0.9 \cdot 0.042210 - 0.9 \cdot 0.0067009 \approx 0.9712$$

(b) Ignoring the fact that the underlying stocks pay dividends gives the following value of the contract

$$\begin{aligned} \Pi(0; X) &= 1.3e^{-rT} + 0.9C^{1.444}(0, 1) - 0.9C^2(0, 1) \\ &= 1.3e^{-rT} + 0.9 \cdot 0.115358 - 0.9 \cdot 0.0275441 \approx 1.0183 \end{aligned}$$

The relative error would have been $(1.0183 - 0.9712)/0.9712 \approx 5\%$.