KTH Mathematics

Exam in SF2701 Financial Mathematics.
Tuesday June 3 2014 8.00-13.00.

Answers and brief solutions.

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1. (a) This exercise can be solved in two ways.

i. Risk-neutral valuation. The martingale measure should satisfy

$$e^{r\Delta t}S_t = qS^u_{t+\Delta t} + (1-q)S^d_{t+\Delta t},$$

so

$$q = \frac{e^{r\Delta t}S_t - S^d_{t+\Delta t}}{S^u_{t+\Delta t} - S^d_{t+\Delta t}}.$$

With numbers

$$q = \frac{e^{0.05\cdot 0.25}40 - 37}{43-37} \approx 0.583856.$$

The price of the option in three months is

$$\Pi(T) = X = \max\{S_T - K\} = \max\{S_T - 40\} = \begin{cases} 3 & \text{if } S_T = 43 \\ 0 & \text{if } S_T = 37 \end{cases}$$

The price at time $t=0$ is then found as

$$\Pi(0) = E^Q\left[\frac{\Pi(T)}{B_T}\right] = e^{-rT}E^Q[X]$$

or with numbers

$$\Pi(0) = e^{-0.05\cdot 0.25}\{0.583856\cdot 3 + (1-0.583856)\cdot 0\} \approx 1.73.$$

ii. Replicating portfolio. The number of stocks in the replicating portfolio is $y = \Delta$, i.e.

$$y = \Delta = \frac{\Delta\Pi}{\Delta S} = \frac{3-0}{43-37} = 0.5.$$

To find the amount of cash x you should have in the bank account solve

$$xe^{r\Delta t} + yS^u_{t+\Delta t} = \Pi^u(t+\Delta t)$$

with numbers

$$xe^{0.05\cdot 0.25} + 0.5\cdot 43 = 3.$$

This yields $x = -e^{0.05\cdot 0.25}18.5$ and the value of the option at time $t=0$ is equal to the value of the replicating portfolio at time $t=0$, that is

$$x + yS_0 = -e^{0.05\cdot 0.25}18.5 + 0.5\cdot 40 \approx 1.73.$$

(b) If we denote by $C(t, K)$ the price at time t of a European call option with strike price K and exercise date T written on the stock, and by $P(t, K)$ the price at time t of a European put option with the same strike price and exercise date as the call, and also having the stock as underlying, then the price of the box spread strategy is given by

$$\Pi_{box}(t) = C(t, K_1) - C(t, K_2) - P(t, K_1) + P(t, K_2).$$

According to put-call parity we have

$$P(t, K) = Ke^{-r(T-t)} + C(t, K) - S(t).$$

Using this we obtain that the price of the box spread strategy is

$$\Pi_{box}(t) = e^{-r(T-t)}(K_2 - K_1).$$

2. (a) We have that

$$\begin{aligned} T &= 6/12 = 1/2 \\ \Delta t &= T/2 = 1/4 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.1331 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.8825 \end{aligned}$$

and the tree for the stock price is therefore

$$\begin{array}{ccc} & & 121.9824 \\ & 107.6491 & \\ 100.0000 & & 95.0000 \\ & 83.8372 & \\ & & 73.9861 \end{array}$$

When the dividend is paid out the jump condition $S_t = S_{t-} - \delta S_{t-}$ is used. Now the option price tree can be computed using

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.4988$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0075}$$

and the result is

$$\begin{array}{ccc} & & 0.0000 \\ & 2.4872 & \\ 9.2712 & & 5.0000 \\ & 16.1628 & \\ & & 26.0139 \end{array}$$

In each node the value is obtained as

$$\max\{100 - S_t, \frac{1}{1.0050}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_t is the current stock price (after dividend payment!), and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 16.1628. The price of the option is thus 9.2712.

(b) The price will decrease, since the option is less likely to be in the money after if the stock pays no dividends (the stock price will be higher).

3. (a) Recall that exchange rates work as assets paying a continuous dividend yield of r_f, where r_f is the foreign interest rate, here the Canadian interest rate.

i. Forwrad prices in general are given by

$$f(t;T,X) = \frac{\Pi_t[X]}{p(t,T)} = \frac{B_t E^Q[S_T/B_T|\mathcal{F}_t]}{p(t,T)}.$$

Using that $B_t = e^{rt}$ (which means that $p(t,T) = e^{-r(T-t)}$) and that the asset pays a continuous dividend yield of δ we obtain that

$$f(t;T,X) = E^Q[S_T|\mathcal{F}_t] = e^{(r-\delta)(T-t)}S_t.$$

Here we get (let $X = S_{0.75}$ be the exchange rate in nine months)

$$f(0;0.75,X) = e^{(0.05-0.04)\cdot 0.75}0.99 \approx 0.99745.$$

ii. Use the Black-Scholes formula with parameters

$$s = S_0 e^{-r_f T} = 0.99e^{-0.04\cdot 0.75}, \quad K = 1, \quad \sigma = 0.1, \quad r = 0.05, \quad T = 0.75$$

to obtain

$$c_{yield} = 0.0320128.$$

Put-call parity for an undelying asset paying a continuous dividen yield is

$$p_{yield}(0) = e^{-rT}K - e^{-\delta T}S_0 + c_{yield}(0).$$

This obtained from the standard Black-Scholes put-call parity by substituting $s = e^{-\delta T}S_0$ everywhere. Using this we obtain

$$p_{yield} = 0.0345.$$

(b) Use the Black -76 formula (which can be obtained from Black-Scholes formula using $s = e^{-r(T-t)}F_t$) with parameters

$$F_0 = 100, \quad K = 103, \quad \sigma = 0.165, \quad , r = 0.05, \quad T = 0.5$$

or the Black-Scholes formula with parameters

$$s = F_0 e^{-rT} = 100e^{-0.05\cdot 0.5}, \quad K = 103, \quad \sigma = 0.165, \quad r = 0.05, \quad T = 0.5.$$

It is important that you do not use the standard Black-Scholes formula here, since the underlying is a commodity which is not ideally traded. The price is

$$c(0) = 3.2890.$$

4. (a) The zero coupon bond prices satisfy

$$p(0,T_i) = e^{-r(0,T_i)}K.$$

Here we have $T_1 = 0.5$ and $T_2 = 1$ and this gives the zero rates

$$r(0,0.5) = 3.0024\%, \quad \text{and} \quad r(0,0.5) = 3.5006\%.$$

Fixed coupon bond prices are computed as

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t,T_i) + Kp(t,T_n)$$

For the two year coupon bond the coupon is $c^2 = 0.02 \cdot 1 \cdot 100 = 2$ and the formula reads

$$96.0874 = 2p(0,1) + (2 + 100)p(0,2).$$

Using that $p(0,1) = 0.9656$ we obtain that $p(0,2) = 0.9231$ and this results in the zero rate

$$r(0,2) = 4.0009\%$$

Finally, the coupon for the three year bond is $c^3 = 0.03 \cdot 1 \cdot 100 = 3$ and the formula reads

$$97.0168 = 3p(0,1) + 3p(0,2) + (3 + 100)p(0,3).$$

Using that $p(0,1) = 0.9656$ and $p(0,2) = 0.9231$, we obtain that $p(0,3) = 0.8869$ and this results in the zero rate

$$r(0,3) = 4.0008\%.$$

(b) The swap rate is set so as to make the value of the fixed and the floating leg equal, i.e.

$$cp(0,1) + (c + K)p(0,2) = K.$$

Using that $c = R_s \cdot 1 \cdot K$, we obtain

$$R_s = \frac{1 - p(0.2)}{p(0,1) + p(0,2)} = \frac{1 - 0.9231}{0.9656 + 0.9231} \approx 0.0407$$

The swap rate is thus $R_s = 4.07\%$.

(c) One can compute the value of the swap as

$$\Pi_{swap} = p_{fixed} - p_{float}.$$

For the fixed coupon bond we have

$$p_{fixed} = (c + K)p(0,T) = (R_s \cdot 1 \cdot K + K)p(0,T) = (R_s + 1)Ke^{-r(0,T)T},$$

since only the final payments remains. With numbers we get

$$p_{fixed} = (0.03 \cdot 1 + 1)10^6 e^{-0.04 \cdot 0.75} \approx 999558.90$$

For the floating rate bond we have

$$\begin{aligned} p_{float} &= (c_i + K)p(0,T_i) = (L(T_{i-1},T_i)\Delta K + K)p(0,T_i) \\ &= (L(T_{i-1},T_i)\Delta + 1)Ke^{-r(0,T_i)T_i}, \end{aligned}$$

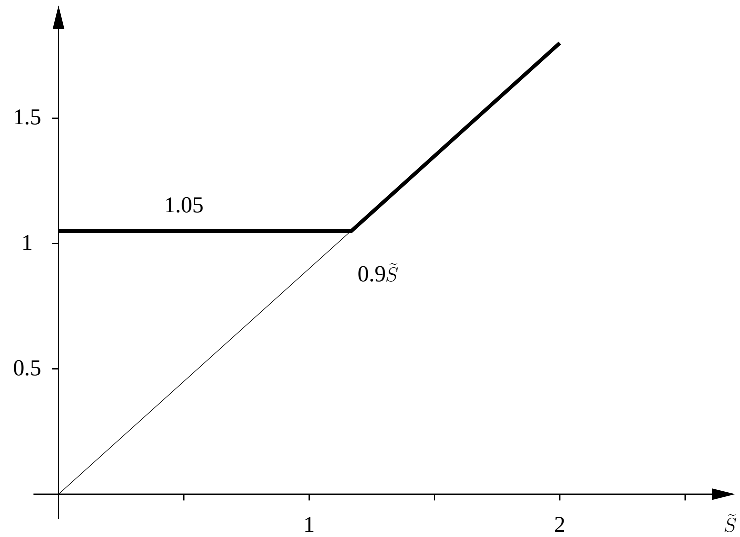
where T_i is the next coupon date. Here this is in three months. With numbers we get

$$p_{float} = (0.035 \cdot 0.5 + 1)10^6 e^{-0.04 \cdot 0.25} \approx 1007375.71$$

The value of the swap to the paty paying floating (and receiving fixed) is therefore

$$\Pi_{swap} = p_{fixed} - p_{float} = 999559 - 1007376 = -7817.$$

5. (a) The contract function of the claim is given by the bold curve in the figure below.



We see that the contract function is piecewise linear, and therefore we know that it can be replicated using a constant portfolio consisting of bonds, call options, and the underlying stock. For the particular claim in this exercise we have

$$\begin{aligned} X &= \max\{1.05, 0.9\widetilde{S}_T\} \\ &= 1.05 + 0.9 \max\left\{\widetilde{S}_T - \frac{1.05}{0.9}, 0\right\} \end{aligned}$$

The replicating portfolio thus consists of 1.05 bonds with maturity T, and 0.9 call options with time of maturity T and strike price 1.05/0.9=7/6. The price of the claim is therefore given by

$$\Pi(0; X) = 1.05e^{-rT} + 0.9C_\delta^{7/6}(0, 1),$$

where $C_\delta^K(t, s)$ denotes the price at time t of a European call option with time of maturity T and strike price K, when the value of the underlying at time t is s and the underlying has a dividend yield δ and volatility σ, and the interest rate is r.

Now using Proposition 16.10 we have that

$$\Pi(0; X) = 1.05e^{-rT} + 0.9C^{7/6}(0, 1 \cdot e^{-\delta(T-t)}),$$

where $C^K(t, s)$ denotes the price at time t of a European call option with time of maturity T and strike price K, when the value of the underlying at time t is s and the underlying has **no** dividend yield, volatility σ, and the interest rate is r. Thus $C^K(t, s)$ is given by the Black-Scholes formula and we obtain

$$\Pi(0; X) = 1.05e^{-0.035 \cdot 5} + 0.9 \cdot 0.055896 \approx 0.9317.$$

(b) For the binary cash-or-nothing call we have

$$
\begin{aligned}
\Pi_t[\mathrm{BCC}_T] &= e^{-r(T-t)} E^Q \left[K I_{\{S_T > K\}} \middle| \mathcal{F}_t \right] \\
&= e^{-r(T-t)} K Q_{t,s} \left(S(T) > K \right) \\
&= e^{-r(T-t)} K \left(1 - Q \left(s e^Z \leq K \right) \right),
\end{aligned}
$$

where $Z \in \mathrm{N}\left((r - \sigma^2/2)(T-t), \sigma^2 (T-t) \right)$. Rewriting a bit gives

$$
\begin{aligned}
\Pi_t[\mathrm{BCC}_T] &= e^{-r(T-t)} K \left[1 - N \left(\frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left\{ \frac{K}{s} \right\} - \left(r - \frac{1}{2}\sigma^2 \right)(T-t) \right\} \right) \right] \\
&= e^{-r(T-t)} K N \left(\frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left\{ \frac{s}{K} \right\} + \left(r - \frac{1}{2}\sigma^2 \right)(T-t) \right\} \right).
\end{aligned}
$$

where we have used the hint to obtain the last equality. We recognize the second half of Black-Scholes formula for the price of a European call option.