KTH Mathematics

Exam in SF2701 Financial Mathematics.
Monday June 8 2015 14.00-19.00.

Answers and brief solutions.

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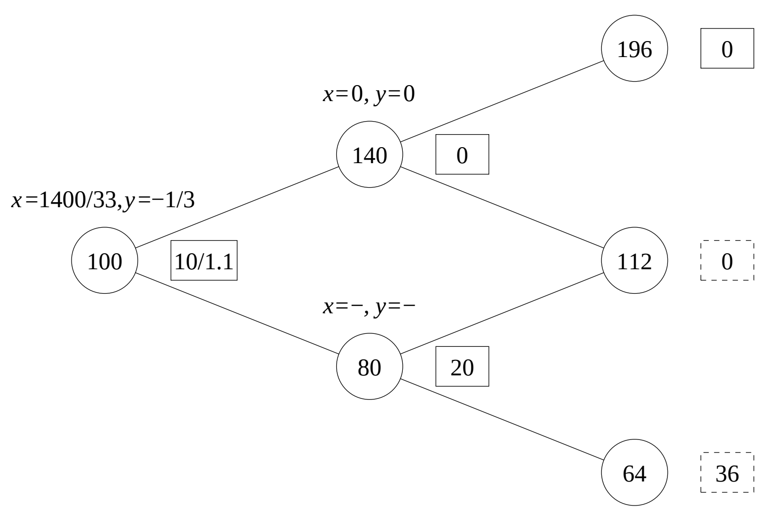
1. (a) Note that the rate given is the one period interest rate, quoted as a single rate. To obtain the replicating portfolio at $t = 0$ we have to solve the following set of equations

$$\begin{cases} 1.1x + y \cdot 140 &= 0, \\ 1.1x + y \cdot 80 &= 20, \end{cases}$$

since regardless of whether the stock price goes up or down the value of the portfolio should equal the value of the option. This yields

$$x = \frac{1400}{33}, \qquad y = -\frac{1}{3}.$$

Using the same method we find the rest of the replicating portfolio strategy and it is shown in the figure below.



Note that since the option is exercised at the node with stock price 80, you will from that node on no longer hold a portfolio.

That the portfolio strategy is self-financing is seen from the following equation

$$1.1 \cdot \frac{1400}{33} - \frac{1}{3} \cdot 140 = 0 + 0 \cdot 140.$$

(b) If we denote by $C(t, S_t)$ the price at time t of a European call option with strike price K and expiry date T written on the stock with price S_t at time t, and by $P(t, S_t)$ the price at time t of a European put option with the same strike price and expiry date as the call, and also having the stock as underlying, then according to put-call parity we have

$$P(t, S_t) = Ke^{-r(T-t)} + C(t, S_t) - S_t.$$

The price of a strap option is thus

$$\Pi_{strap}(t) = 2C(t, S_t) + P(t, S_t) = 3C(t, S_t) + Ke^{-r(T-t)} - S_t.$$

Using $t = 0$, $S_0 = 80$, $K = 80$, $r = 0.02$, $\sigma = 0.3$, and $T = 0.5$ we find that the value of the strap option is

$$\Pi_{strap}(0) = 3 \cdot 7.1294 + 80e^{-0.02 \cdot 0.5} - 80 = 20.5923,$$

i.e. \$20.5923.

2. (a) We have that

$$\begin{aligned} T &= 4/12 = 1/3 \\ \Delta t &= T/2 = 1/6 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.0851 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.9216 \end{aligned}$$

and the tree for the stock price is therefore

		111.8520
	103.0822	
100.0000		95.0000
	87.5515	
		80.6870

Now the option price tree can be computed using

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.5103,$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0050}$$

and the result is

		11.8520
	8.5076	
4.3194		0.0000
	0.0000	
		0.0000

In each node the value is obtained as

$$\max\{S_{t-} - 100, \frac{1}{1.0050}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_{t-} is the current stock price **just before dividend payment**, and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 8.5076. The price of the option is thus 4.3194.

(b) If we denote the option price by Π we have that

$$\Delta = \frac{\partial \Pi}{\partial s} \approx \frac{\Delta \Pi}{\Delta s}.$$

This gives us

$$\Delta = \frac{11.8520 - 0}{111.8520 - 95.00} \approx 0.7033$$

if we end up in the node with stock price 103.0822, and

$$\Delta = \frac{0 - 0}{95.00 - 80.6870} = 0,$$

if we end up in the node with stock price 87.5515. Note that since early exercise is optimal in the node with stock price 103.0822 the delta computed does not represent part of a replicating portfolio in that node.

3. (a) i. The implied volatility is the volatility which if inserted into the Black-Scholes formula (together with the other parameters of the the option) yields the current option price. If we insert $S_0 = 336$, $K = 350$, $r = 0.01$, $T = 0.5$, and $\sigma = 0.2$ (and $t = 0$) into the Black-Scholes formula we get

$$c(0) = 13.7735.$$

If we instead use $\sigma = 0.3$ we obtain

$$c(0) = 23.1818.$$

The implied volatility is thus 20%.

ii. The number of stock you should add to your portfolio is given by the delta of the option. We have that

$$\Delta = \Phi[d_1(t, s)]$$

for a European call option in the standard Black-Scholes framework. Here Φ is the cumulative distribution function for the $N(0, 1)$ distribution and

$$d_1(t, s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}.$$

Inserting $S_0 = 336$, $K = 350$, $r = 0.01$, $T = 0.5$, and $\sigma = 0.2$ (and $t = 0$) we obtain

$$\Delta = 0.4276.$$

(b) Use the Black -76 formula (which can be obtained from Black-Scholes formula using $s = e^{-r(T-t)}F_t$) with parameters

$$F_0 = 2332, \quad K = 2350, \quad \sigma = 0.166, \quad r = 0.02, \quad T = 0.5$$

or the Black-Scholes formula with parameters

$$s = F_0 e^{-rT} = 2332e^{-0.02 \cdot 0.5}, \quad K = 2350, \quad \sigma = 0.166, \quad r = 0.02, \quad T = 0.5.$$

The price of the corresponding call option is therefore

$$c_{fut}(0) = 99.7926$$

The put-call parity for futures options reads

$$p_{fut}(0) = e^{-rT}K - e^{-rT}F_0 + c_{fut}(0).$$

This can be obtained from the standard Black-Scholes put-call parity by substituting $s = e^{-rT}F_0$ everywhere. Using put-call parity we obtain

$$p_{fut}(0) = 117.6135.$$

4. (a) The zero coupon bond prices satisfy

$$p(0, T_i) = e^{-r(0,T_i)}K.$$

Here we have $T_1 = 0.5$ and this gives the zero rate

$$r(0, 0.5) = 2.0101\%.$$

and $T_2 = 1$ yielding the zero rate

$$r(0, 1) = 2.5010\%.$$

Fixed coupon bond prices are computed as

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t, T_i) + Kp(t, T_n)$$

For the two year coupon bond the coupon is $c^2 = 0.02 \cdot 1 \cdot 100 = 2$ and the formula reads

$$98.0106 = 2p(0, 1) + (2 + 100)p(0, 2).$$

Using that $p(0, 1) = 0.9753$ we obtain that $p(0, 2) = 0.9417$ and this results in the zero rate

$$r(0, 2) = 3.0000\%.$$

(b) When a forward rate agreement is set up the borrowing (lending) rate should be chosen as the current forward rate in order for the value of the forward rate agreement to be zero. We have the following relationship between forward rates and spot rates

$$r(t, T)(T - t) = r(t, S)(S - t) + f(t; S, T)(T - S).$$

Thus

$$f(t; S, T)(T - S) = r(t, T)(T - t) - r(t, S)(S - t)$$

and we get the one year forward rate for the second year is

$$f(0; 1, 2)(2 - 1) = r(0, 2) \cdot 2 - r(0, 1) \cdot 1 = 3.0000 \cdot 2 - 2.5010 \cdot 1 = 3.4990\%,$$

The LIBOR rate $L(0; S, T)$ is the same rate as the forward rate $f(0; S, T)$, only quoted as a simple rate. We thus have

$$1 + L(0; S, T)(T - S) = e^{f(0;S,T)(T-S)}$$

In this exercise we need the LIBOR rate for the second year and this is obtained from

$$1 + L(0;1,2) \cdot 1 = e^{f(0;1,2)\cdot 1},$$

so

$$L(0;1,2) = e^{f(0;1,2)\cdot 1} - 1 = e^{0.03490} - 1 \approx 0.035609$$

The borrowing (lending) rate should thus be set to 3.56% per annum with annual compounding.

(c) We know the following

$$\begin{aligned} K_{SEK} &= 8.000.000 \\ K_{USD} &= 1.000.000 \\ c_{SEK} &= 8.000.000 \cdot 0.02 = 160.000 \\ c_{USD} &= 1.000.000 \cdot 0.04 = 40.000 \end{aligned}$$

where $K_{\cdot}$ denotes the principals in the respective currencies, and $c_{\cdot}$ denotes the payments in the respective currencies. The value of the swap is given by

$$\Pi_{swap}(t) = p^{SEK}_{fixed}(t) - X_{SEK/USD}(t)p^{USD}_{fixed}(t)$$

where $X_{SEK/USD}(t)$ is the exchange rate between SEK and USD, which is currently 8.4 SEK/USD. In general a fixed coupon bond price is given by

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t,T_i) + Kp(t,T_n).$$

Here we have that

$$\begin{aligned} p^{SEK}_{fixed}(t) &= \frac{160.000}{1.02} + \frac{160.000 + 8.000.000}{1.02^2} = 8.000.000 \\ p^{USD}_{fixed}(t) &= \frac{40.000}{1.04} + \frac{40.000 + 1.000.000}{1.04^2} = 1.000.000. \end{aligned}$$

The current value of the swap to the Swedish company is thus

$$\Pi_{swap}(t) = 8.000.000 - 8.4 \cdot 1.000.000 = -400.000 SEK.$$

5. (a) Note that a forward contract with forward price f, and delivery date T, can be seen as a T-claim with payoff

$$X_{forward} = S_T - f.$$

Now we have that

$$X_{forward} = S_T - f = \max\{S_T - f, 0\} - \max\{f - S_T, 0\},$$

which means that the payoff of the forward contract can be replicated by buying a European call option with strike price equal to the forward price and exercise date equal to the delivery date of the forward contract, and selling the corresponding put option.

(b) For the T-claim $X = \phi(S_T) = \max\{S_T^p - K, 0\}$ we have

$$\Pi_0[X] = e^{-rT} E^Q \left[\max\{S_T^p - K, 0\}\right]$$

Recall that $S_T = S_0 e^Z$ where $Z \in N\left((r - \sigma^2/2)T, \sigma\sqrt{T}\right)$. This means that

$$S_T^p = S_0^p e^{pZ} = S_0^p e^Y,$$

where $pZ = Y \in N\left(p(r - \sigma^2/2)T, p\sigma\sqrt{T}\right)$. Now compare this to the distribution of a stock price process X of a stock paying a continuous dividend yield of δ:

$$X_T = X_0 e^U,$$

where $U \in N\left((r - \delta - \bar{\sigma}^2/2)T, \bar{\sigma}\sqrt{T}\right)$. We then see that if we choose

$$\begin{aligned} \bar{\sigma} &= p\sigma, \\ (r - \delta - \bar{\sigma}^2/2)T &= p(r - \sigma^2/2)T, \end{aligned}$$

i.e.

$$\begin{aligned} \bar{\sigma} &= p\sigma, \\ \delta &= (1-p)r - \frac{\sigma^2}{2}p(p-1). \end{aligned}$$

then the price of the powered call option is given as the price of call option written on a stock paying a continuous dividend yield of δ, starting from $X_0 = S_0^p$ and with volatility $p\sigma$. Using that this price can be obtained from the ordinary Black-Scholes formula replacing S_0 by $e^{-\delta(T-t)}X_0$. If we let $c(t, s, K, T, r, \sigma)$ denote the standard Black-Scholes price at time t of a European call option with exercise price K and expiry date T, when the current price of the underlying is s, the interest rate is r, and the volatility of the underlying is σ, then we have that

$$c^p(0, s, K, T, r, \sigma) = c(0, e^{-\delta T}s^p, K, T, r, p\sigma)$$

where

$$\delta = (1-p)r - \frac{\sigma^2}{2}p(p-1).$$