KTH Mathematics

Exam in SF2701 Financial Mathematics.
Friday June 3 2016 08.00-13.00.

Answers and brief solutions.

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1. (a) This exercise can be solved in two ways.

i. Risk-neutral valuation. The martingale measure should satisfy
$$e^{r\Delta t}S_t = qS^u_{t+\Delta t} + (1-q)S^d_{t+\Delta t},$$
so
$$q = \frac{e^{r\Delta t}S_t - S^d_{t+\Delta t}}{S^u_{t+\Delta t} - S^d_{t+\Delta t}}.$$
With numbers
$$q = \frac{e^{0.05\cdot 0.25}100 - 80}{120-80} \approx 0.531446.$$
The price of the option in three months is
$$\Pi(T) = X = \max\{K - S_T\} = \max\{100 - S_T\} = \begin{cases} 0 & \text{if } S_T = 120 \\ 20 & \text{if } S_T = 80 \end{cases}$$
The price at time $t=0$ is then found as
$$\Pi(0) = E^Q\left[\frac{\Pi(T)}{B_T}\right] = e^{-rT}E^Q[X]$$
or with numbers
$$\Pi(0) = e^{-0.05\cdot 0.25}\{0.531446\cdot 0 + (1-0.531446)\cdot 20\} \approx 9.2547.$$

ii. Replicating portfolio. The number of stocks in the replicating portfolio is $y=\Delta$, i.e.
$$y = \Delta = \frac{\Delta\Pi}{\Delta S} = \frac{0-20}{120-80} = -0.5.$$
To find the amount of cash x you should have in the bank account solve
$$xe^{r\Delta t} + yS^u_{t+\Delta t} = \Pi^u(t+\Delta t)$$
with numbers
$$xe^{0.05\cdot 0.25} - 0.5\cdot 120 = 0.$$
This yields $x = e^{-0.05\cdot 0.25}60$ and the value of the option at time $t=0$ is equal to the value of the replicating portfolio at time $t=0$, that is
$$x + yS_0 = e^{-0.05\cdot 0.25}60 - 0.5\cdot 100 \approx 9.2547.$$

(b) i. If we denote by $C(t,K)$ the price at time t of a European call option with strike price K and exercise date T written on the stock, and by $P(t,K)$ the price at time t of a European put option with the same strike price and exercise date as the call, and also having the stock as underlying, then according to put-call parity we have

$$P(t,K) = e^{-r(T-t)}K + C(t,K) - S(t).$$

The arbitrage price of the syntethic stock is therefore

$$P(t,K) - C(t,K) = e^{-r(T-t)}K - S(t) = e^{-0.02*0.5}85 - 80 = 4.1542.$$

ii. The market price of the conversion strategy is

$$S(t) + P_{market}(t,K) - C_{market}(t,K) = 80 + 9.3 - 5.0 = 84.3$$

whereas the arbitrage price is

$$S(t) + P_{arb}(t,K) - C_{arb}(t,K) = 80 + 4.1542 = 84.1542$$

which means that a profit of

$$84.1542 - 84.3 = -0.1458$$

is made (so a loss).

2. (a) We have that

$$\begin{aligned} T &= 6/12 = 1/2 \\ \Delta t &= T/2 = 1/4 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.1331 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.8825 \end{aligned}$$

and the tree for the futures price is therefore

		128.4025
	113.3148	
100.0000		100.0000
	88.2497	
		77.8801

Now the option price tree can be computed using

$$q = \frac{1-d}{u-d} \approx 0.4688,$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0050}$$

and the result is

		0.0000
	0.0000	
6.2107		0.0000
	11.7503	
		22.1199

In each node the value is obtained as

$$\max\{100 - S_t, \frac{1}{1.0050}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where S_t is the current stock price, and P^u and P^d is the price of the option if the stock price goes up and down, respectively. Early exercise will be optimal in the node with option price 11.7503. The price of the option is thus 6.2107.

(b) If we denote the option price by Π we have that

$$\Delta = \frac{\partial \Pi}{\partial f} \approx \frac{\Delta \Pi}{\Delta f}.$$

This gives us

$$\Delta = \frac{0 - 11.7503}{113.3148 - 88.2497} \approx -0.4688.$$

3. (a) Use the Black-Scholes formula with parameters

$$s = S_0 e^{-\delta T} = 100e^{-0.05 \cdot 0.75}, \quad K = 97, \quad \sigma = 0.25, \quad r = 0.03, \quad T = 0.75$$

to obtain

$$c_{index} = 8.9992.$$

(b) If we denote the price of a derivative written on the underlying stock by Π we have by definition that

$$\Delta = \frac{\partial \Pi}{\partial s}.$$

For a European call option in the standard Black-Scholes framework this yields

$$\Delta_{call} = \Phi[d_1(t, s)].$$

Here Φ is the cumulative distribution function for the $N(0, 1)$ distribution and

$$d_1(t, s) = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}.$$

Let $C(t, S_t)$ denote the price of a call option given by the standard Black-Scholes formula. Using the definition of delta and the expression for delta for a European call option together with the fact that $c_{index} = C(t, e^{-\delta(T-t)} S_t)$ we obtain that

$$\Delta_{index} = e^{-\delta(T-t)} \Phi[d_1(t, e^{-\delta(T-t)} S_t)].$$

With the same parameters as in exercise 3a we obtain

$$\Delta_{index} = e^{-0.05 \cdot 0.75} 0.5713 = 0.5503.$$

(c) If we denote by $C(t, S_t, K, T)$ the price at time t of a European call option with strike price K and exercise date T written on the stock, and by $P(t, S_t, K, T)$ the price at time t of a European put option with the same strike price and exercise date as the call, and also having the stock as underlying, then according to standard put-call parity we have

$$P(t, S_t, , K, T) = Ke^{-r(T-t)} + C(t, S_t, K, T) - S_t.$$

Now if we replace S_t by $S_t * e^{-q(T-t)}$ we get put-call parity for assets paying a continuous dividend yield

$$p_{index} = Ke^{-r(T-t)} + c_{index} - e^{-q(T-t)} S_t.$$

Solving for for q we obtain

$$q = -\frac{1}{T-t} \ln \left(\frac{Ke^{-r(T-t)} + c_{index} - p_{index}}{S_t} \right)$$

Here we have that $t = 0$ and

$$S_0 = 512, \quad K = 505, \quad r = 0.03, \quad T = 0.25, \quad c_{index} = 32, \quad p_{index} = 28.$$

Inserting these numbers we obtain

$$q = 0.05327.$$

4. (a) i. Zero coupon bond prices satisfy

$$p(0, T_i) = e^{-r(0,T_i)T_i} K.$$

Here we have $T_1 = 0.5$, $K = 100$, and $r(0, 0.5) = 2.0\%$ and this yields the bond price

$$p(0, 0.5) = e^{-0.02 \cdot 0.5} 100 = 99.0050.$$

ii. Fixed coupon bond prices are computed as

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t, T_i) + K p(t, T_n).$$

For the two year coupon bond the coupon is $c^2 = 0.02 \cdot 0.5 \cdot 100 = 1$ and the formula reads

$$p^2_{fixed}(0) = 1p(0, 0.5) + 1p(0, 1) + 1p(0, 1.5) + (1 + 100)p(0, 2).$$

Using that $p(0, T_i) = e^{-r(0,T_i)T_i}$ this results in the coupon bond price

$$p^2_{fixed}(0) = 1e^{-r(0,0.5)0.5} + 1e^{-r(0,1)1} + 1e^{-r(0,1.5)1.5} + (1 + 100)e^{-r(0,2)2} = 98.0517.$$

iii. For the three year coupon bond the coupon is $c^3 = 0.03 \cdot 1 \cdot 100 = 3$ and the formula reads

$$p^3_{fixed}(0) = 3p(0, 1) + 3p(0, 2) + (3 + 100)p(0, 3).$$

Again using that $p(0, T_i) = e^{-r(0,T_i)T_i}$ this results in the coupon bond price

$$p^3_{fixed}(0) = 3e^{-r(0,1)1} + 3e^{-r(0,2)2} + (3 + 100)e^{-r(0,3)3} = 97.8064.$$

(b) We know the following

$$\begin{aligned} K_{GBP} &= 7.000.000 \\ K_{USD} &= 10.000.000 \\ c_{USD} &= 10.000.000 \cdot 0.03 = 300.000 \end{aligned}$$

where $K_\cdot$ denotes the principals in the respective currencies, and $c_\cdot$ denotes the payments in the respective currencies. The value of the swap is given by

$$\Pi_{fixed-for-float}(t) = X_{GBP/USD}(t) p^{USD}_{fixed}(t) - p^{GBP}_{float}(t)$$

where $X_{GBP/USD}(t)$ is the exchange rate between GBP and USD, which is currently 0.69 GBP/USD. In general a fixed coupon bond price is given by

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t, T_i) + K p(t, T_n).$$

Here we have that

$$p^{USD}_{fixed}(t) = \frac{300.000}{1.04} + \frac{300.000 + 10.000.000}{1.04^2} = 9.811.390, 533.$$

For the floating rate bond we have that the value is equal to the principal since one payment has just been made.

The current value of the swap to the party paying British pounds and receiving dollars is thus

$$\Pi_{fixed-for-float}(t) = 0.69 \cdot 9.811.390,533 - 7.000.000 = -230.140,5325\ GBP.$$

5. (a) Recall that in a complete market it should be possible to replicate all claims.

The market described in the exercise is incomplete. To show this we need to find a claim that can not be replicated. Let Z denote a random variable which equals u with probability p, 1 with probability q, and d with probability $1-p-q$. If the market was complete it would be possible to solve the following set of equations for every function $\phi(z)$ (then $\mathbf{h} = (x, y)$ would be a replicating portfolio).

$$\begin{cases} x(1+r) + ysu &= \phi(u), \\ x(1+r) + ys &= \phi(1), \\ x(1+r) + ysd &= \phi(d). \end{cases}$$

Let ϕ be given by

$$\phi(z) = \begin{cases} K, & \text{if } z = u \text{ or } z = 1, \\ 2K, & \text{if } z = d. \end{cases}$$

Solving for x and y using the first two equations gives $x = K/(1+r)$ and $y = 0$, but this does not satisfy the third equation. This means that there are claims which cannot be replicated in this model and therefore the model is not complete.

(b) i. Using the Black-Scholes formula we see that the price of the at-the-money call option is given by

$$c(S(T_0), T - T_0, S(T_0), \sigma) = S_{T_0} N(d_1) - e^{-r(T-T_0)} S_{T_0} N(d_2).$$

Here N is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-T_0}} \left\{ \left(r + \frac{1}{2}\sigma^2 \right)(T - T_0) \right\}, \\ d_2 &= d_1 - \sigma\sqrt{T-T_0}. \end{aligned}$$

This means that

$$c(S(T_0), T - T_0, S(T_0), \sigma) = \beta S_{T_0},$$

where

$$\beta = N(d_1) - e^{-r(T-T_0)} N(d_2)$$

is a positive constant (since the option price is always positive, except possibly at maturity). This menas that the payoff X can be written as

$$X = \max\{\beta S_{T_0} - K_0, 0\} = \beta \max\left\{ S_{T_0} - \frac{K_0}{\beta}, 0 \right\}.$$

We now recognize the payoff as the constant β times the payoff from a European call option written on S, with maturity T_0, and strike price K_0/β.

The price can thus be obtained using the Black-Scholes formula with parameters

$$s = S_0, \quad K = K_0/\beta, \quad T = T_0,$$

and r, and σ as before, and multiplying the result by β.

ii. Inserting the numbers you will get $K_0 = 9.4134$, $\beta = 0.06371$, and the price $\Pi(0, X) = 0.001334$.