KTH Mathematics

Exam in SF2701 Financial Mathematics.
Monday June 5 2017 08.00-13.00.

Answers and brief solutions.

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1. (a) We have that

$$\begin{aligned} T &= 6/12 = 1/2 \\ \Delta t &= T/2 = 1/4 \\ u &= 1.05 \\ d &= 0.95 \end{aligned}$$

and the tree for the stock price is therefore

		110.25
	105.00	
100.00		99.75
	95.00	
		90.25

The price of the option in six months is

$$\Pi_T = \max\{K - S_T, 0\} = \max\{100 - S_T, 0\} = \begin{cases} 0 & \text{if } S_T = 110.25 \\ 0.25 & \text{if } S_T = 99.75 \\ 9.75 & \text{if } S_T = 90.25 \end{cases}$$

Now the option price tree can be computed using

$$q = \frac{e^{r\Delta t} - d}{u - d} \approx 0.5753,$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0075}$$

and the result is

		0.0000
	0.1054	
1.8529		0.2500
	4.2528	
		9.7500

In each node the value is obtained as

$$\frac{1}{1.0075}(q \cdot P^u + (1-q) \cdot P^d)$$

where P^u and P^d is the price of the option if the stock price goes up and down, respectively. The price of the option is thus 1.8529.

(b) i. We have that

$$\max\{K - C(T, S_T), 0\} = K - C(T, S_T) + \max\{C(T, S_T) - K, 0\}. \quad (1)$$

Now the price at time t of any T-claim X is given by

$$\Pi_t[X] = e^{-r(T-t)} E^Q [X|\mathcal{F}_t]$$

The price of the payoff on the left hand side of (1) is the price of a European put on a call, P_{call}, and the price of the payoff on the right hand side can be written

$$\begin{aligned}&\Pi_0[K - C(T, S_T) + \max\{C(T, S_T) - K, 0\}] = \\ &\Pi_0[K] - \Pi_0[C(T, S_T)] + \Pi_0[\max\{C(T, S_T) - K, 0\}] = \\ &e^{-rT}K - C(0, S_0) + C_{call}\end{aligned}$$

where C_{call} denotes the price of a European call on a call. To see that

$$\Pi_0[C(T, S_T)] = C(0, S_0)$$

use that

$$\begin{aligned}\Pi_0[C(T, S_T)] &= e^{-rT} E^Q [C(T, S_T)] \\ &= e^{-rT} E^Q \left[e^{-r(T_1-T)} E^Q [\max\{S_{T_1} - K_1, 0\}|\mathcal{F}_T]\right] \\ &= e^{-rT_1} E^Q [\max\{S_{T_1} - K_1, 0\}] \\ &= C(0, S_0)\end{aligned}$$

where we have used the tower property that the smallest σ-algebra always wins.

2. (a) We have that

$$\begin{aligned}T &= 4/12 = 1/3 \\ \Delta t &= T/2 = 1/6 \\ u &= e^{\sigma\sqrt{\Delta t}} \approx 1.0632 \\ d &= e^{-\sigma\sqrt{\Delta t}} \approx 0.9406\end{aligned}$$

and the tree for the futures price is therefore

		56.5145
	53.1576	
50.0000		50.0000
	47.0300	
		44.2364

Now the option price tree can be computed using

$$q = \frac{1-d}{u-d} \approx 0.4847,$$

and the discount factor

$$\frac{1}{e^{r\Delta t}} \approx \frac{1}{1.0084}$$

and the result is

$$\begin{array}{ccc} & & 8.5145 \\ & 5.1576 & \\ 2.9704 & & 2.0000 \\ & 0.9613 & \\ & & 0.0000 \end{array}$$

In each node the value is obtained as

$$\max\{F_t - 48, \frac{1}{1.0084}(q \cdot P^u + (1-q) \cdot P^d)\}$$

where F_t is the current futures price, and P^u and P^d is the price of the option if the futures price goes up and down, respectively. Early exercise will be optimal in the node with option price 5.1576. The price of the option is thus 2.9704.

(b) It makes no difference what the underlying is, what is used above is that the futures price process is a Q-martingale.

3. (a) Recall that exchange rates work as assets paying a continuous dividend yield of r_f, where r_f is the foreign interest rate, here the interest rate in the United states.

i. Forward prices in general are given by

$$f(t;T,X) = \frac{\Pi_t[X]}{p(t,T)} = \frac{B_t E^Q[S_T/B_T|\mathcal{F}_t]}{p(t,T)}.$$

Using that $B_t = e^{rt}$ (which means that $p(t,T) = e^{-r(T-t)}$) and that the asset pays a continuous dividend yield of δ we obtain that

$$f(t;T,X) = E^Q[S_T|\mathcal{F}_t] = e^{(r-\delta)(T-t)}S_t.$$

Here we get (let $X = S_{2/3}$ be the exchange rate in eight months (2/3 of a year))

$$f(0;2/3,X) = e^{(0.04-0.06)\cdot 2/3}0.89 \approx 0.8782.$$

ii. Use the Black-Scholes formula with parameters

$$s = S_0e^{-r_fT} = 0.89e^{-0.06\cdot 2/3}, \quad K = 0.88, \quad \sigma = 0.1, \quad r = 0.04, \quad T = 2/3$$

to obtain

$$c_{yield} = 0.0270.$$

(b) If we denote the price of a derivative written on the underlying stock by Π we have by definition that

$$\Delta = \frac{\partial \Pi}{\partial s}.$$

For a European call option in the standard Black-Scholes framework this yields

$$\Delta_{call} = \Phi[d_1(t,s)].$$

Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$d_1(t,s) = \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}.$$

If we denote by $C(t,S_t,K,T)$ the price at time t of a European call option with strike price K and exercise date T written on the stock, and by $P(t,S_t,K,T)$

the price at time t of a European put option with the same strike price and exercise date as the call, and also having the stock as underlying, then according to standard put-call parity we have

$$P(t, S_t, , K, T) = Ke^{-r(T-t)} + C(t, S_t, K, T) - S_t.$$

Using this we obtain that

$$\Delta_{put} = \Phi[d_1(t, s)] - 1.$$

There is also a result saying that if you have sold a derivative and want to make your portfolio delta neutral then you should add a number of stocks equalt to the delta of the derivative. Using

$$s = 230, \quad K = 225, \quad \sigma = 0.20, \quad r = 0.01, \quad T = 1/2$$

we obtain

$$\Delta_{put} = \Phi[d_1(0, s)] - 1 = \Phi[0.2615] - 1 = -0.3969.$$

You should therefore sell 0.40 stocks in order to make your portfolio delta neutral.

4. (a) Start by finding the current term structure of zero rates. Zero coupon bond prices satisfy

$$p_K(0, T_i) = e^{-r(0,T_i)T_i} K.$$

Here we have $T_1 = 1$, $K = 100$ and $p_{100}(0, 1) = 99.00$ and this gives the zero rate

$$r(0, 1) = 1.0050\%.$$

Fixed coupon bond prices are computed as

$$p_{fixed}(t) = \sum_{i=1}^{n} c_i p(t, T_i) + Kp(t, T_n)$$

For the two year coupon bond the coupon is $c^2 = 0.03 \cdot 1 \cdot 100 = 3$ and the formula reads

$$102.92 = 3p(0, 1) + (3 + 100)p(0, 2).$$

Using that $p(0, 1) = 0.9900$ we obtain that $p(0, 2) = 0.9704$ and this results in the zero rate

$$r(0, 2) = 1.5029\%$$

Finally, the coupon for the three year bond is $c^3 = 0.02 \cdot 1 \cdot 100 = 2$ and the formula reads

$$99.98 = 2p(0, 1) + 2p(0, 2) + (2 + 100)p(0, 3).$$

Using that $p(0, 1) = 0.9900$ and $p(0, 2) = 0.9704$, we obtain that $p(0, 3) = 0.$ and this results in the zero rate

$$r(0, 3) = 2.0002\%.$$

When a forward rate agreement is set up the borrowing (lending) rate should be chosen as the current forward rate in order for the value of the forward rate agreement to be zero. We have the following relationship between forward rates and spot rates

$$r(t,T)(T-t) = r(t,S)(S-t) + f(t;S,T)(T-S).$$

Thus

$$f(t;S,T)(T-S) = r(t,T)(T-t) - r(t,S)(S-t)$$

and we get that the one year forward rate for the third year is

$$f(0;2,3)(3-2) = r(0,3)\cdot 3 - r(0,2)\cdot 2 = 2.0002\cdot 3 - 1.5029\cdot 2 = 2.9949\%,$$

The LIBOR rate $L(0;S,T)$ is the same rate as the forward rate $f(0;S,T)$, only quoted as a simple rate. We thus have

$$1 + L(0;S,T)(T-S) = e^{f(0;S,T)(T-S)}$$

In this exercise we need the LIBOR rate for the third year and this is obtained from

$$1 + L(0;2,3)\cdot 1 = e^{f(0;2,3)\cdot 1},$$

so

$$L(0;2,3) = e^{f(0;2,3)\cdot 1} - 1 = e^{0.029949} - 1 \approx 0.030402$$

The borrowing (lending) rate should thus be set to 3.04% per annum with annual compounding.

(b) One can compute the value of the swap as

$$\Pi_{swap} = p_{float} - p_{fixed}.$$

For the fixed coupon bond we have

$$p_{fixed} = cp(0,0.5) + (c+K)p(0,1),$$

where $c = 0.017 * 0.5 * 1.000.000 = 8500$. The only thing unknown to us in the formula for the price for the fixed coupon bond is $p(0,0.5)$. This can be found from the swap rate, since the swap rate is set so as to make the value of the fixed and the floating leg equal, i.e.

$$(c+K)p(0,0.5) = K.$$

Using that $c = R_s \cdot \Delta T \cdot K$, we obtain

$$p(0,0.5) = \frac{1}{1 + R_s\cdot\Delta T} = \frac{1}{1 + 0.01\cdot 0.5} \approx 0.9950$$

The price of the fixed coupon bond is therefore (recall that $p(0,1) = 0.99$)

$$p_{fixed} = 8500p(0,0.5) + (8500 + 1000000)p(0,1) = 1006872.71.$$

For the floating rate bond we have that the value is equal to the principal since a coupon payment has just been made so $p_{float} = 1.000.000$, and therefore the value of the swap is

$$\Pi_{swap} = p_{float} - p_{fixed} = 1.000.000 - 1006872.71 = -6872.71.$$

5. (a) i. Note that a forward contract with forward price f, and delivery date T, written on a stock, S, can be seen as a T-claim with payoff

$$X_{forward} = S_T - f.$$

The payoff of the European put option on the stock with maturity T and a strike price f is

$$X_{put} = \max\{f - S_T, 0\},$$

The total payoff at time T is therefore

$$X_{port} = X_{forward} + X_{put} = S_T - f + \max\{f - S_T, 0\} = \max\{S_T - f, 0\}.$$

ii. From exercise 5(a)i we see that the payoff of the portfolio is equal to that of a European call option with the strike price f and maturity T. Therefore we have that

$$\Pi(X_{port}) = \Pi(X_{forward}) + \Pi(X_{put}) = \Pi(X_{call}).$$

Since the price of $X_{forward}$ is zero when the forward contract is set up, we see that the price of the put and call options are the same when the portfolio (and forward contract) is set up.

(b) Note that we can write the payoff from the gap option as

$$\phi(s) = \begin{cases} s - K_2, & \text{if } s > K_2, \\ 0, & \text{otherwise.} \end{cases} + \begin{cases} K_2 - K_1, & \text{if } s > K_2, \\ 0, & \text{otherwise.} \end{cases}$$

Now the first payoff is that of a European call option with strike price K_2, the price of which can be obtained from the standard Black-Scholes formula.

If we introduce

$$I\{x > K\} = \begin{cases} K & \text{if } x > K, \\ 0 & \text{otherwise} \end{cases}$$

The second payoff function can be written $(K_2 - K_1)I\{s > K_2\}$ The price of $X = (K_2 - K_1)I\{S_T > K_2\}$ at time t is

$$\Pi_t = e^{-r(T-t)}E^Q\left[(K_2 - K_1)I\{S_T > K_2\}|\,\mathcal{F}_t\right] = e^{-r(T-t)}(K_2 - K_1)E^Q\left[I\{S_T > K_2\}|\,\mathcal{F}_t\right]$$

Now, recall that $S_T = S_t e^Z$, where $Z \in N((r - \sigma^2/2)(T-t), \sigma^2(T-t))$. The expectation can now be computed and the result is

$$Q(S_T > K_2) = N[d_2(t)],$$

with $K = K_2$ (for more details see the solution to exercise 5 b of the exam given in June 2014). The price of the gap option is therefore

$$\begin{aligned} \Pi_{gap} &= S_0\Phi[d_1] - e^{-rT}K_2\Phi[d_2] + e^{-rT}(K_2 - K_1)\Phi[d_2] \\ &= S_0\Phi[d_1] - e^{-rT}K_1\Phi[d_2] \end{aligned}$$

where Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T}}\left\{\ln\left(\frac{S_0}{K_2}\right) + \left(r + \frac{1}{2}\sigma^2\right)T\right\}, \\ d_2 &= d_1 - \sigma\sqrt{T}. \end{aligned}$$